Optics

R.J. Marks II Class Notes

Fourier: Rose-Hulman Institute of Technology (1971)

Statistical: Texas Tech University (1976)

OPTICS

3-15-72 (WED)

2-9, 2-12, 2-13

3-16-72 (THURS)

$\omega = 2\pi f = \frac{2\pi}{T}$ TIME

$k = \frac{2\pi}{\lambda}$ SPACE

$f(t) = \int_{-\infty}^{\infty} F(f) e^{jft} df$

$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \ni \omega = 2\pi f$

TRAVELING WAVES

$e^{j(\omega t - kx)}$ $\ni \omega t - kx$ IS PHASE FACTOR

$V_p = \frac{\omega}{k}$ = PHASE VELOCITY

FOR SINUSOIDAL-STEADY STATE

USE e^{-jkx}

$f(x) = \int_{-\infty}^{\infty} F(k) e^{jkx} dk$

FOR TWO DIMENSIONAL TRANSFORM

$f(x,y) = \int_{-\infty}^{\infty}\int F(k_x, k_y) e^{j(k_x x + k_y y)} dk_x dk_y$

MULTIPLY BOTH SIDES BY $e^{j\omega t}$

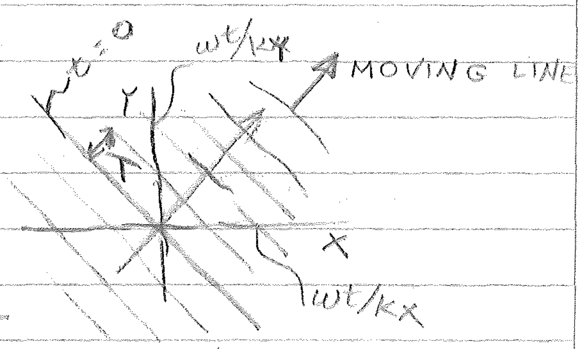
$e^{j(\omega t - k_x x - k_y y)}$ $\Rightarrow$ PHASE PHACTOR

DIFFERENTIATE, & SET TO 0

$\Rightarrow \omega = k_x V_x + k_y V_y$

SET TO CONSTANT

$k_x x + k_y y = \omega t$



$\frac{\omega t}{kx}$ DETERMINES DIRECTION $\Leftarrow$

(2-1) a) $\delta(ax, by) = \frac{1}{|ab|}\delta(x,y)$

$f(0,0) = \iint_{-\infty}^{\infty} f(x,y)\,\delta(x,y)\,dx\,dy$

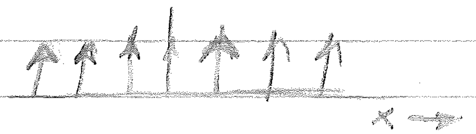
$f(\xi,\eta) = \iint_{-\infty}^{\infty} f(x,y)\,\delta(x-\xi,\,y-\eta)\,dx\,dy$

INTRODUCE CHANGE OF VARIABLES

$\xi = ax,\ \eta = by \Rightarrow dx\,dy = \frac{1}{|ab|}\,d\xi\,d\eta$

b) $comb(ax)\,comb(ay) = \frac{1}{|ab|}\sum_{n=-\infty}^{\infty}\delta(x-\frac{n}{a},\,y-\frac{m}{b})$

$comb(ax) = \sum_{n=-\infty}^{\infty}\delta(ax-n)$



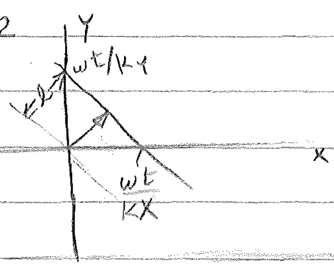
$= \frac{1}{|a|}\sum_{n=-\infty}^{\infty}\delta(x-n/a)$

$comb\ by = \frac{1}{|b|}\sum_{m=-\infty}^{\infty}\delta(y-\frac{m}{b})$

$\therefore comb\ ax\ comb\ by = \frac{1}{|ab|}\sum_{n=-\infty}^{\infty}\delta(x-\frac{n}{a})\sum_{m=-\infty}^{\infty}\delta(y-\frac{m}{b})$

$= \frac{1}{|ab|}\sum_{m,n=-\infty}^{\infty}\delta(x-\frac{n}{a},\,y-\frac{m}{b})$

3-17-72



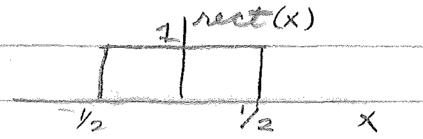
$l = \frac{1}{\sqrt{\frac{K_x^2}{\omega^2t^2}+\frac{K_y^2}{\omega^2t^2}}} = \frac{\omega t}{\sqrt{K_x^2+K_y^2}}$

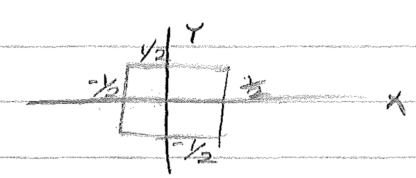
$v = \frac{dl}{dt} = \omega/K \Rightarrow \vec{K} = K_x + K_y$

$V_x, V_y > V$

TURN IN 2-5 MONDAY

(2-2)(a) $\mathcal{F}\{rect(x)\,rect(y)\} = sinc(f_x)\,sinc(f_y)$





(CONT)

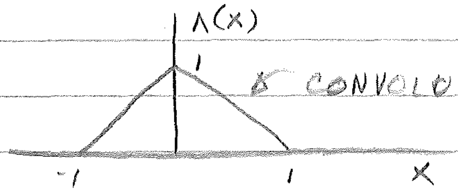
$$\mathcal{F}(\;) = \int_{-\infty}^{\infty}\!\!\int \mathrm{rect}(x)\,\mathrm{rect}(y)\, e^{-j2\pi(f_x x + f_y y)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \mathrm{rect}(x)\, e^{-j2\pi f_x x}\,dx \cdot \int_{-\infty}^{\infty} \mathrm{rect}(y)\, e^{-j2\pi f_y y}\,dy$$

$$= \mathcal{F}\{\mathrm{rect}(x)\}\,\mathcal{F}\{\mathrm{rect}(y)\}$$

$$= \frac{\sin(\pi f_x)}{\pi f_x}\,\frac{\sin(\pi f_y)}{\pi f_y}$$

b) $\mathcal{F}\{\Lambda(x)\,\Lambda(y)\} = \mathrm{sinc}^2(f_x)\,\mathrm{sinc}^2(f_y)$



CONVOLUTION OF THE RECT FUNCTIONS

$$\Rightarrow \mathcal{F}\{\Lambda(x)\,\Lambda(y)\} = \mathrm{sinc}^2(f_x)\,\mathrm{sinc}^2(f_y)$$

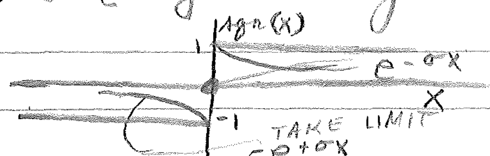
c) $\mathcal{F}\{\delta(x,y)\} = \int_{-\infty}^{\infty}\!\!\int \delta(x,y)\, e^{-j2\pi(f_x x + f_y y)}\,dx\,dy = 1$

$$\mathcal{F}^{-1}\{1\} = \delta(x,y) = \int_{-\infty}^{\infty}\!\!\int (1)\, e^{j2\pi(x f_x + f_y y)}\,df_x\,df_y$$

$$= \delta(x,y)$$

$$\mathcal{F}\{\delta(x-\xi,\, y-\eta)\} = e^{-j2\pi(f_x \xi + f_y \eta)}$$

d) $\mathcal{F}\{\mathrm{sgn}(x)\,\mathrm{sgn}(y) = \frac{1}{j\pi f_x}\,\frac{1}{j\pi f_y}$



$$\mathcal{F}\{\mathrm{sgn}(x)\,\mathrm{sgn}(y)\} = \mathcal{F}\{\mathrm{sgn}(x)\}\cdot\mathcal{F}\{\mathrm{sgn}(y)\}$$

$$= \int_{-\infty}^{\infty} \mathrm{sgn}(x)\, e^{-j2\pi f_x x}\,dx \cdot \int_{-\infty}^{\infty} \mathrm{sgn}(y)\, e^{-j2\pi f_y y}\,dy$$

NOW

$$\mathcal{F}\{e^{\mp}\,\mathrm{sgn}\,x\} = -\int_{-\infty}^{0} e^{\sigma x} e^{-j2\pi f_x x}\,dx + \int_{0}^{\infty} e^{-\sigma x} e^{-j2\pi f_x x}\,dx$$

$$= \frac{-1}{\sigma - j2\pi f_x} + \frac{1}{\sigma + j2\pi f}$$

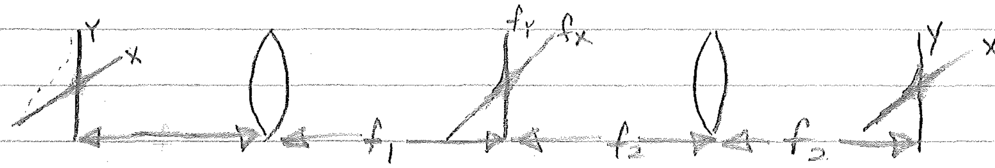
TAKE LIMAT AS $\sigma \to 0$

$$\Rightarrow \mathcal{F}\{\mathrm{sgn}\,x\} = \frac{1}{j\pi f_x}$$

$$\mathcal{F}\{\mathrm{sgn}(x)\,\mathrm{sgn}(y)\} = \frac{1}{j\pi f_y}\,\frac{1}{j\pi f_x}$$

(2-6) $\mathcal{F}_A\{g\} = G_A(f_X, f_Y) = \frac{1}{a}\iint_{-\infty}^{\infty} g(\xi,\eta)\, e^{-j\frac{2\pi}{a}(f_X\xi + f_Y\eta)}\, d\xi\, d\eta$

$\mathcal{F}_B\{g\} = \frac{1}{b}\iint_{-\infty}^{\infty} g(\xi,\eta)\, e^{-j\frac{2\pi}{b}(f_X\xi + f_Y\eta)}\, d\xi\, d\eta$



$$\mathcal{F}_B[\mathcal{F}_A\{g(x,y)\}] = \frac{1}{b}\iint_{-\infty}^{\infty} \exp\left[-j\frac{2\pi}{b}(\xi f_X + \eta f_Y)\right] G_A(f_X, f_Y)\, df_X\, df_Y$$

$$= \frac{1}{ab}\iint_{-\infty}^{\infty} \exp\left[-j\frac{2\pi}{b}(\xi f_X + \eta f_Y)\right]\left[\iint_{-\infty}^{\infty} g(x,y)\exp\left[-j\frac{2\pi}{a}(f_X x + f_Y y)\right] dx\, dy\right] df_X\, df_Y$$

$$= \frac{1}{ab}\iint_{-\infty}^{\infty} g(x,y)\left\{\iint_{-\infty}^{\infty} \exp\left[j2\pi\left[\left(\frac{x}{a}+\frac{\xi}{b}\right)f_X + \left(\frac{y}{a}+\frac{\eta}{b}\right)f_Y\right]\right]\right\} df_X\, df_Y\, dx\, dy$$

$$= \frac{1}{ab}\iint_{-\infty}^{\infty} g(x,y)\, \delta\left(\frac{x}{a}+\frac{\xi}{b}, \frac{y}{a}+\frac{\eta}{b}\right) dx\, dy$$

$\frac{x}{a} = u \quad \frac{y}{a} = v \Rightarrow dx = a\,du;\ dy = a\,dv$

$$\mathcal{F}_B[\mathcal{F}_A\{g(x,y)\}] = \frac{1}{ab}\iint_{-\infty}^{\infty} g(au, av)\, \delta\left(u + \frac{\xi}{b}, v + \frac{\eta}{b}\right) a^2\, du\, dv$$

$$= \frac{a}{b}\, g\left(-\frac{a}{b}\xi, -\frac{a}{b}\eta\right)$$

IF $a = b = 1$: $\mathcal{F}(\mathcal{F}\{g\}) = g(-x,-y)$

(2-3)(a) SPECIALIZED CASE OF (2-6)

(b) $\mathcal{F}\{g(x,y)\, h(x,y)\} = \mathcal{F}\{g(x,y)\} \circledast \mathcal{F}\{h(x,y)\}$

$\circledast \Rightarrow$ COMPLEX CONVOLUTION

$$\equiv g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi,\eta)\, h(x_2-\xi, y_2-\eta)\, d\xi\, d\eta$$

LET $h(x,y) = \iint_{-\infty}^{\infty} H(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X\, df_Y$

$\leftarrow \mathcal{F}^{-1}$ FORM OF $h(x,y)$

$\therefore g(x,y)\, h(x,y) = g(x,y)\iint_{-\infty}^{\infty} H(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X\, df_Y$

$$\therefore \mathcal{F}\{g(x,y)\, h(x,y)\} = \iint_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi(\xi_X x + \xi_Y y)} \left\{\iint_{-\infty}^{\infty} H(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X\, df_Y\right\} dx\, dy$$

$$= \iint_{-\infty}^{\infty} H(f_X, f_Y)\left\{\iint_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi[x(-f_X + \xi_X) + y(-f_Y + \xi_Y)]}\, dx\, dy\right\} df_X\, df_Y$$

3-20-72 (MON)

$$\mathcal{F}\{g(x,y)\,h(x,y)\} = \int_{-\infty}^{\infty}\!\!\int \underbrace{g(x,y)\,e^{-j2\pi[x(q_x-f_x)+y(q_y-f_y)]}\,dx\,dy}_{G(q_x-f_x,\,q_y-f_y)} \int_{-\infty}^{\infty}\!\!\int H(f_x,f_y)\,df_x\,df_y$$

$$= \int_{-\infty}^{\infty}\!\!\int H(f_x,f_y)\,G(q_x-f_x,\,q_y-f_y)\,df_x\,df_y$$

$$= \mathcal{F}\{g\} \circledast \mathcal{F}\{h\}$$

(2-12 AND 2-13 DUE)

(2-11) $\nabla_{XY} = \left| \dfrac{\int_{-\infty}^{\infty}\int g(x,y)\,dx\,dy}{g(0,0)} \right|$ = NORMALIZED EQUIVALENT AREA

$\nabla_{f_x f_y} = \left| \dfrac{\int_{-\infty}^{\infty}\int G(f_x,f_y)\,df_x\,df_y}{G(0,0)} \right|$ = EQUIV. B.W.

$$\nabla_{xy}\nabla_{f_x f_y} = \frac{\left|\int\int g(x,y)\,dx\,dy\right|\left|\int\int G(f_x f_y)\,df_x\,df_y\right|}{|G(0,0)|\,|g(0,0)|}$$

NOTE $G(f_x,f_y) = \int_{-\infty}^{\infty}\int g(x,y)\,e^{-j2\pi(f_x x + f_y y)}\,dx\,dy$

THUS: $G(0,0) = \int_{-\infty}^{\infty}\int g(x,y)\,dx\,dy$

ALSO $g(x,y) = \int_{-\infty}^{\infty}\int G(f_x,f_y)\,e^{j2\pi(f_x x + f_y y)}\,df_x\,d_{f_y}$

THUS: $g(0,0) = \int_{-\infty}^{\infty}\int G(f_x,f_y)\,df_x\,df_y$

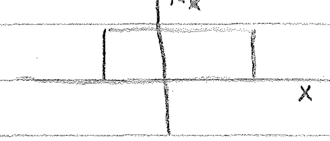
$\Rightarrow \nabla_{xy}\nabla_{f_x f_y} = 1$ (HEISENBURG'S UNCERTAINTY PRINCIPLE)

3-22-72 (WED)

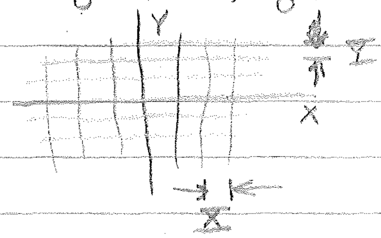
SAMPLING THEM.

BAND LIMITED (SPECTRUM IS BOUNDED IN k_x, k_y SPACE)

(FINITE ENERY SIGNAL)



SAMPLED $g(x,y)$; $g_s(x,y) = comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right) g(x,y)$



IN ORDER TO COMPUTE SPECTRUM, USE COMPLEX CONV:

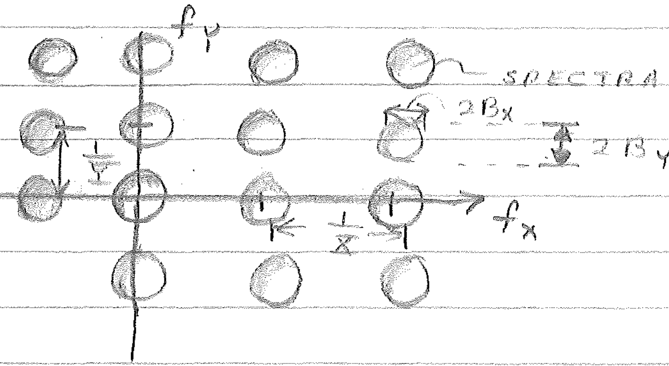
$$G_s(f_x,f_y) = \mathcal{F}\left\{comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right)\right\} \circledast G(f_x,f_y)$$

$$= XY\, comb(Xf_x)\, comb(Yf_y) \circledast G(f_x,f_y)$$

$$= \sum_{m,n}\sum \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right) \circledast G(f_x,f_y)$$

$$G_s(q_x,q_y) = \sum_{m,n} \int\int_{-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}\right) G(q_x - f_x, q_y - f_y)\, df_x\, df_y$$

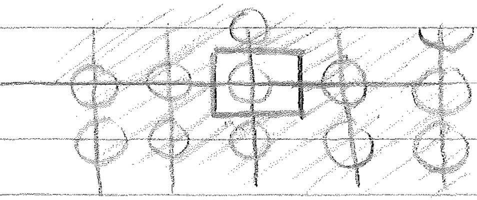
$$= \sum_{mn} G\left(q_x - \frac{n}{X}, q - \frac{m}{Y}\right)$$



$$\frac{1}{X} \geq 2B_x \Rightarrow X \leq \frac{1}{2B_x}$$

$$\frac{1}{Y} \geq 2B_y \Rightarrow Y \leq \frac{1}{2B_y}$$

PASS SPECTRA THRU FILTER TO GET BACK 1 SPECTRA



$O(f_x, f_y) = H(f_x, f_y)\, G_s(f_x, f_y)$

$H(f_x, f_y) = \text{rect}\left(\frac{f_x}{2B_x}\right) \text{rect}\left(\frac{f_y}{2B_y}\right)$ LOW-PASS FILTER

IN SPACE DOMAIN

$g(x,y) = h(f_x, f_y) \circledast g(f_x, f_y)$

$= \left[\text{comb}\left(\frac{x}{X}\right) \text{comb}\left(\frac{y}{Y}\right) g(x,y)\right] \circledast h(x,y)$

$h(x,y) = 4B_x B_y \,\text{sinc}(2B_x x)\, \text{sinc}(2B_y y)$

$\Rightarrow g(x,y) = 4B_x B_y X Y \sum_{m,n} g(nX, mY)\, \text{sinc}[2B_x(x - nX)]$
$\text{sinc}[2B_y(y - mY)]$

3-24-72 (FRI)

$\nabla \times H = \epsilon_0 \frac{\delta E}{\delta t}$

$\nabla \times E = -\mu_0 \frac{\delta H}{\delta t}$

SINUSOIDAL STEADY-STATE

$\bar{E} = R_e(\bar{E} e^{-j\omega t})$

$\bar{H} = R_e(\bar{H} e^{-j\omega t})$

$\Rightarrow \nabla \times \bar{H} = j\omega \epsilon_0 \bar{E}$

$\nabla \times \bar{E} = j\omega \mu_0 H$

$\nabla \times \nabla \times \bar{H} = j\omega\epsilon_0 \nabla \times \bar{E} = \omega^2 \mu_0 \epsilon_0 \bar{H}$

$\nabla \times \nabla \times H = -\nabla^2 H + \nabla(\nabla \bar{H}) = -\nabla^2 \bar{H}$

$\therefore -\nabla^2 \bar{H} = \omega^2 \mu_0 \epsilon_0 H = \frac{\omega^2}{c^2} = k^2 \bar{H}$; VECTOR HELMHOLTZ EQN.

WORKING IN ONE DIMENSION $k = 2\pi/\lambda$

LET U BE A RECT COMP OF $\bar{H}$ (OR $\bar{E}$)

$\Rightarrow \nabla^2 U + k^2 V = 0$; SCALAR HELMHOLTZ EQN.

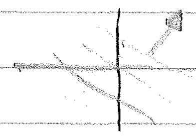
DIGRESSION ON WAVE TYPES IN VARIOUS COORDINATES

1) RECTANGULAR

$$\frac{\delta^2 U}{\delta X^2} + \frac{\delta^2 U}{\delta Y^2} + \frac{\delta^2 V}{\delta Z^2} + K^2 V = 0$$

$\Rightarrow U(x,Y,Z) = A e^{\pm j(k_x X + k_Y Y + k_Z Z)}$ $\Rightarrow$ PLANE WAVE TRAVELING IN $\pm k$ DIRECTION

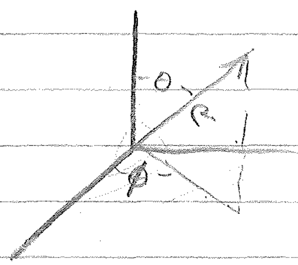
$\bar{k} = k_x a_x + k_Y a_Y + k_Z a_Z$



PLANE: $k_x X + k_Y Y + k_Z Z$ = CONSTANT ; (PLANE)

$k^2 = \left(\frac{\omega}{c}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 = k_x^2 + k_Y^2 + k_Z^2 \Rightarrow |\bar{k}| = 2\pi/\lambda$

2) SPHERICAL COORDINATES



$$\nabla^2 = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta}{\delta r}\right)$$

$$\therefore \nabla^2 V + k^2 U = 0 = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta V}{\delta r}\right) + k^2 V$$

$\Rightarrow V = \frac{1}{r} e^{\pm jkr}$ (K IS NOT A VECTOR)

FOR CONSTANT PHASE kr = CONST

$\Rightarrow$ SPHERE OF RADIUS = $\frac{\text{CONST}}{k}$

$$Ve^{-j\omega t} = \frac{1}{r} e^{j(\pm kr - \omega t)}$$

$$\frac{d}{dt}(\pm k_r - \omega t) = \pm k \frac{dV}{dt} - \omega = 0$$

$$\Rightarrow \pm \frac{dr}{dt} = \frac{\omega}{k} = c$$

PLUS SIGN $\Rightarrow$ SPHERICALLY DIVERGING FROM ORIGIN

MINUS SIGN $\Rightarrow$ " CONVERGING " "

SOMMERFIELD-RADIATION CONDITION:

FOR OUTGOING WAVES, WE MUST SATISFY THE CONDITION

$$\lim_{R\to\infty} R\left(\frac{\delta V}{\delta R} - jkV\right)=0 \quad \text{ON A LARGE SPHERE}$$

FOR INCOMING WAVES (OR REFLECTED WAVES @ ∞) WE MUST HAVE:

$$\lim_{R\to\infty} R\left(\frac{\delta V}{\delta R} + jkV\right)=0$$

EXAMPLE:

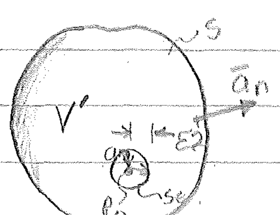
(3-1)

$$V=\frac{A}{R}e^{-jkR}, \quad \frac{\delta V}{\delta R} = \pm jk\frac{A}{R}e^{\pm jkR} - \frac{A}{R^2}e^{\pm jkR}$$

$$\therefore \frac{\delta V}{\delta R}\pm jkV = \frac{-Ae^{\pm jkR}}{R^2}$$

$$\lim_{R\to\infty} R\left(\frac{-A}{R^2}e^{\pm jkR}\right) = \lim\left(\frac{-A}{R}e^{\pm jkR}\right)=0$$

GREEN'S THEM:



$$\iiint_{V'}(G\nabla^2U - U\nabla^2G)\,dV = \oint\!\!\oint\left(G\frac{\delta U}{\delta n} - U\frac{\delta G}{\delta n}\right)ds$$

LET $\nabla^2 G + k^2 G = 0$ (HEMHOLTZ)

$\nabla^2 U + k^2 U = 0$

LET G BE A GREEN'S FUNCTION HAVING A POLE (OR SINGULARITY) AT P_0. (SUCH AS $\frac{1}{r}e^{\pm jkr}$)

ie G BEHAVES AS A SPHERICALLY OUTGOING WAVE WHOSE SOURCE IS @ P_0.

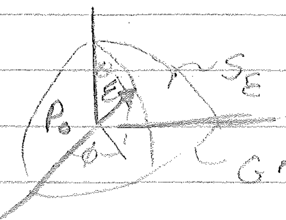
THEN $G\nabla^2U - U\nabla^2G = -kGU + k^2GU = 0$

$$\oint\!\!\oint\left(G\frac{\delta U}{\delta n} - U\frac{\delta G}{\delta n}\right)ds_T = \iint_S\left(G\frac{\delta U}{\delta n} - U\frac{\delta G}{\delta n}\right)dS + \iint\left(G\frac{\delta U}{\delta n} - U\frac{\delta G}{\delta n}\right)dS = 0$$

NEAR P_0, $G \sim \frac{1}{r}$, THUS $\frac{\delta G}{\delta r} \sim -\frac{1}{r^2}$ BUT IN THE NEGATIVE DIRECTION $\therefore \frac{\delta G}{\delta r} \sim \frac{1}{r^2}$

$$\therefore \iint_S \left(G \frac{\delta U}{\delta n} - U \frac{\delta G}{\delta n}\right) dS = \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \cdot \varepsilon^2 \left(\frac{-1}{\varepsilon} \frac{\delta U}{\delta r} - \frac{U(P_0)}{\varepsilon^2}\right)}_{A}$$

$\varepsilon \to 0$



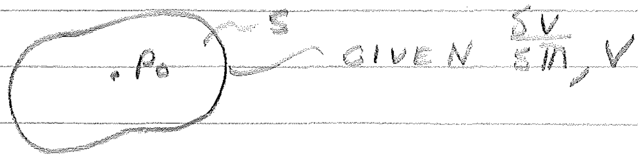
$dA = d\phi \sin\theta \, d\theta$

$$\lim_{\varepsilon \to 0} A = \int_0^{2\pi} \int_0^{\pi} d\phi \int_0^{\pi} \varepsilon \, \theta \, d\theta \, (-U(P_0))$$
$$= -4\pi \, U(P_0)$$

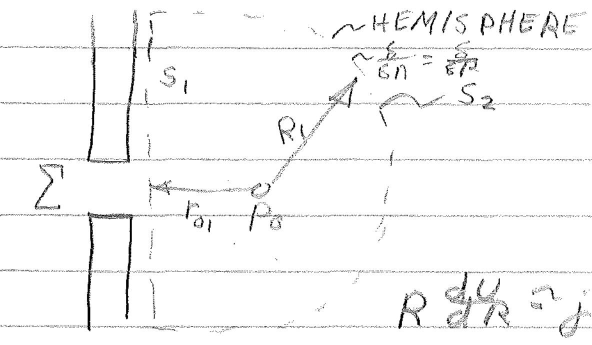
THUS:

$$U(P_0) = \frac{1}{4\pi} \iint_S \left(G \frac{\delta U}{\delta n} - U \frac{\delta G}{\delta n}\right) dS$$

G IS AN OUTGOING GREEN'S FUNCTION WITH SOURCE @ P_0



IT APPEARS THAT KNOWLEDGE OF BOTH U AND $\frac{\delta U}{\delta n}$ IS NECESSARY. NOT SO! (AS WE'LL SEE)



$R \frac{\delta G}{\delta R} \simeq jkGR$ FOR $R \to \infty$

(FROM SOMMERFIELD)

$$\lim_{R \to \infty} R\left(\frac{\delta G}{\delta R} \overset{\text{OUTGOING}}{+} jkG\right) = 0$$

$$R \frac{dU}{dR} \simeq jkUR$$

$$\oiint S = \iint S_1 + \iint S_2$$

CONSIDER: $\iint_{S_2} \left(G \frac{\delta U}{\delta n} - U \frac{\delta G}{dn}\right) dS$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \, R^2 \left(G \frac{dU}{dR} + U \frac{\delta G}{\delta R}\right)$$

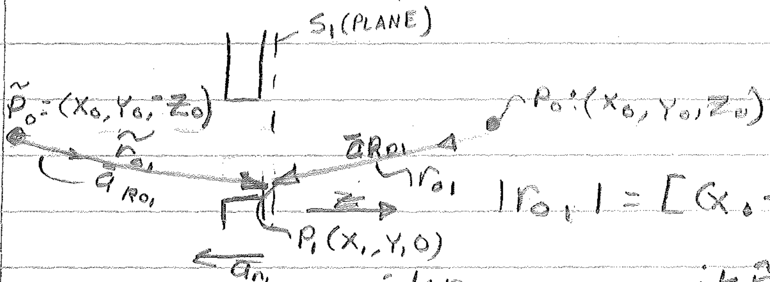
$\to 0$ as $R \to \infty$

BY VIRTUE OF THE SOMMERFIELD RAD CONDITION

$\therefore U(P_0) = \frac{1}{4\pi} \iint_{S_1} \left(G \frac{\delta U}{\delta n} - U \frac{\delta G}{\delta n} \right) dS$

3-27-72 (MONDAY)

① $U(P_0) = \frac{1}{4\pi} \iint_{S_1} \left(G \frac{\delta U}{\delta n} - U \frac{\delta G}{\delta n} \right) dS$



$|r_{01}| = \left[(X_0 - X_1)^2 + (Y_0 - Y_1)^2 + Z_0^2 \right]^{\frac{1}{2}}$

$G(P_1) = \frac{1}{r_{01}} e^{jkr_{01}} - \frac{1}{\tilde{r}_{01}} e^{jk\tilde{r}_{01}}$

ON THE SURFACE: $G(P_1) = 0$

NOW

$$\left. \frac{\delta G(P_1)}{\delta n} \right|_{\text{ON SCREEN}} = \bar{a}_n \cdot \nabla G(P_1)$$
$$= \bar{a}_n \cdot \nabla \left(\frac{1}{r_{01}} e^{jkr_{01}} \right)$$
$$- \bar{a}_n \nabla \left(\frac{1}{\tilde{r}_{01}} e^{jk\tilde{r}_{01}} \right)$$

NOW $\nabla \left(\frac{1}{r_{01}} e^{jkr_{01}} \right) = \bar{a}_{r_{01}} \frac{\delta}{\delta r_{01}} \left(\frac{1}{r_{01}} e^{jkr_{01}} \right)$

$$= \bar{a}_{r_{01}} \left(jk - \frac{1}{r_{01}} \right) \left(\frac{1}{r_{01}} e^{jkr_{01}} \right)$$

$$\Rightarrow \left. \frac{\delta G(P_1)}{\delta n} \right|_{\text{ON SURF}} = \bar{a}_n \cdot \bar{a}_{R01} \left(jk - \frac{1}{r_{01}} \right) \left(\frac{1}{r_{01}} e^{jkr_{01}} \right)$$
$$- \bar{a}_n \cdot \tilde{a}_{R01} \left(jk - \frac{1}{\tilde{r}_{01}} \right) \left(\frac{1}{\tilde{r}_{01}} e^{jk\tilde{r}_{01}} \right)$$

NOW $\bar{a}_n \cdot \bar{a}_{R01} = -\bar{a}_n \cdot \tilde{\bar{a}}_{r01}$

$$\Rightarrow \left. \frac{\delta G(P_1)}{\delta n} \right|_{\text{ON SCREEN}} = 2 \bar{a}_n \cdot \bar{a}_{R01} \left(jk - \frac{1}{r_{01}} \right) \left(\frac{1}{r_{01}} e^{jkr_{01}} \right)$$

SUBSTITUTING BACK IN ①

$$U(P_0) = \frac{1}{4\pi} \int_{S_1} 2 \bar{a}_n \cdot \bar{a}_{R01} \left(\frac{1}{r_{01}} - jk \right) \frac{e^{jkr_{01}}}{r_{01}} U(P_1) \, dS$$

$K = \frac{2\pi}{\lambda} \sim 10{,}000 \, \mathring{A}$ (FOR LASORS) $= 10^{-6} M$

$\rightarrow K \gg \frac{1}{r_{01}}$

3-3) 2ND APP. DERIV

$$U(P_0) = \frac{-j2k}{4\pi} \int_S \bar{a}_n \cdot \bar{a}_{R01} \frac{1}{r_{01}} e^{jkr_{01}} U(P_1) \, dS$$

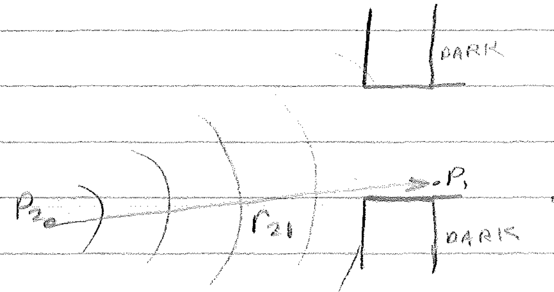
A TWO DIMENSIONAL CONVOLUTION

$$U(P_0) = \int h(P_0 - P_1) \, U(P_1) \, dS$$

NOTE REPEPROCITY

HOMEWORK: 3-3 - GETS OTHER PART FROM ①

CONSIDER SPECIAL CASE (3-2)



$U(P_1) = A e^{jkr_{21}}$

$$\Rightarrow U(P_0) = \frac{A}{j\lambda} \int_\Sigma \bar{a}_n \cdot \bar{a}_{R_0} \frac{1}{r_{21} r_{01}} e^{jk(r_{01}+r_{21})} dS$$

3-29-72 (WED)

ANGULAR SPECTRUM

TO SOLVE HELMHOLTZ EQUATION USING FOURIER TRANS.

$$\nabla^2 U + k^2 U = 0 = \frac{\delta^2 U}{\delta x^2} + \frac{\delta U^2}{\delta y^2} + \frac{\delta^2 U}{\delta z^2} + k^2 U \Rightarrow k^2 = \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda}\right)^2$$

LET $U(x,y,z) = \int_{-\infty}^{\infty}\int A(f_x, f_y)\, e^{j2\pi[f_x x + f_y y + z(\frac{1}{\lambda^2} - f_x^2 - f_y^2)^{1/2}]} df_x df_y$

$$\nabla^2 U = \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)\left[\int_{-\infty}^{\infty}\int A(f_x,f_y) e^{j2\pi[f_x x + f_y y + z(\frac{1}{\lambda^2} - f_x^2 - f_y^2)^{1/2}]} df_x df_y\right]$$

$$= \int_{-\infty}^{\infty}\int A(f_x,f_y)\left[-(2\pi)^2 f_x^2 - (2\pi)^2 f_y^2 - (2\pi)^2\left(\frac{1}{\lambda^2} - \cancel{f_x^2} - f_y^2\right)\right] e^{j2\pi[\quad]} df_x df_y$$

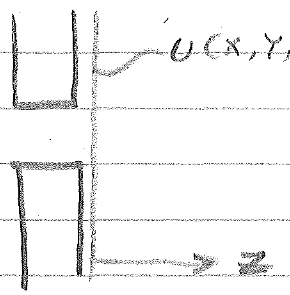
$$= -k^2 \int_{-\infty}^{\infty}\int A(f_x,f_y) e^{j2\pi[\quad]} df_x df_y$$

$= -k^2 U(x,y,z) \Rightarrow$ SOLUTION OF HEMHOLTZ'S EQUATION

$$U(x,y,0) = \int_{-\infty}^{\infty}\int A(f_x,f_y) e^{j2\pi[f_x x + f_y y]} df_x df_y$$

$$\Rightarrow A(f_x,f_y) = \int_{-\infty}^{\infty}\int U(x,y,0) e^{-j2\pi[f_x x + f_y y]} dx dy$$

$\leftarrow$ ANGULAR SPECTRUM



$U(x,y,0)$; $\mathcal{F}[U(x,y,0)] = A(f_x,f_y)$

NOTE THAT IF $A(f_x,f_y)$ IS THE X-FORM OF THE INPUT, THEN $A(f_x,f_y)e^{j2\pi(\frac{1}{\lambda^2} - f_x^2 - f_y^2)^{1/2}}$ IS THE F X-FORM OF THE OUTPUT

OUTPUT:

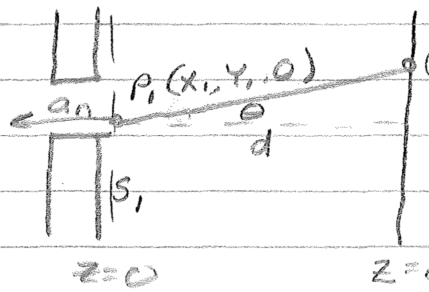
$$U(x,y) = \int_{-\infty}^{\infty}\int \underbrace{A(f_x,f_y) e^{j2\pi z(\frac{1}{\lambda^2} - f_x^2 - f_y^2)}}_{A_0} e^{j2\pi(f_x x + f_y y)} df_x df_y$$

SYSTEM TRANSFORM FUNCTION

$$\frac{A_0(f_X,f_Y)}{A_{IN}(f_X,f_Y)} = e^{j2\pi z\left(\frac{1}{\lambda^2}-f_X^2-f_Y^2\right)^{\frac{1}{2}}}$$

REMEMBER:

$$U(P_0)=\frac{1}{j\lambda}\int_S \frac{\exp\left(jk\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+d^2}\right)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+d^2}}\,\bar{a}_n\cdot\bar{a}_{r01}\cdot U(x,y)\,dx_1\,dy_1$$



$$\bar{a}_n\cdot\bar{a}_{r01}=\frac{d}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+d^2}}$$

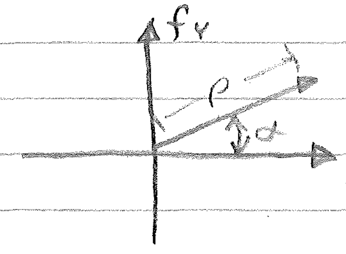
$$\Rightarrow U(P_0)=\frac{d}{j\lambda}\int_S \frac{\exp\left(jk\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+d^2}\right)}{(x_2-x_1)^2+(y_2-y_1)^2+d^2}$$

$$=\underbrace{\frac{d}{j\lambda}\,\frac{\exp\left(jk\sqrt{x^2+y^2+d^2}\right)}{x^2+y^2+z^2}}_{=\mathcal{F}^{-1}\left[e^{j2\pi d\left(\frac{1}{\lambda^2}-f_X^2-f_Y^2\right)^{\frac{1}{2}}}\right]} \circledast U(P_1)$$

TO SHOW $\mathcal{F}^{-1}\left[e^{j2\pi\left(\frac{1}{\lambda^2}-f_X^2-f_Y^2\right)^{\frac{1}{2}}}\right] = \frac{-jd}{\lambda}\,\frac{e^{jk(x^2+y^2+d^2)^{\frac{1}{2}}}}{(x^2+y^2+z^2)}$

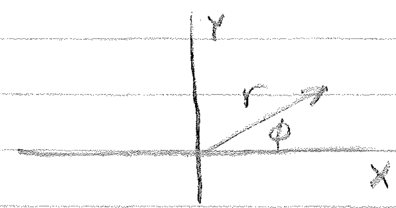
FOR $d^2+x^2+y^2 \gg \lambda$

$$\mathcal{F}^{-1}[H(f_X,f_Y)]=\int_{-\infty}^{\infty}\!\!\int e^{j2\pi d\left[\frac{1}{\lambda^2}-f_X^2-f_Y^2\right]^{1/2}}e^{j2\pi(f_Xx+f_Yy)}\,df_X\,df_Y$$



$f_X=\rho\cos\alpha$

$f_Y=\rho\sin\alpha$



$x=r\cos\phi$

$Y=r\sin\phi$

$df_X\,df_Y=\rho\,d\rho\,d\alpha$

$$\mathcal{F}^{-1}[H(f_X,f_Y)]=\int_0^{\infty}e^{j2\pi d\left[\frac{1}{\lambda^2}-\rho^2\right]}\rho\,d\rho\underbrace{\int_0^{2\pi}e^{j2\pi\rho r\cos(\alpha-\phi)}\,d\alpha}_{2\pi J_0(2\pi r\rho)}$$

J_0 IS BESSEL FUNCTION OF ZERO ORDER

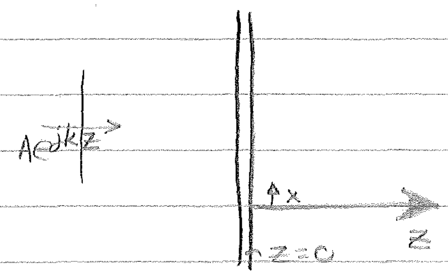
$$\mathcal{F}^{-1}[H(f_X,f_Y)]=2\pi\int_0^{\infty}e^{jkd(1-\lambda^2\rho^2)^{1/2}}J_0(2\pi r\rho)\,d\rho$$

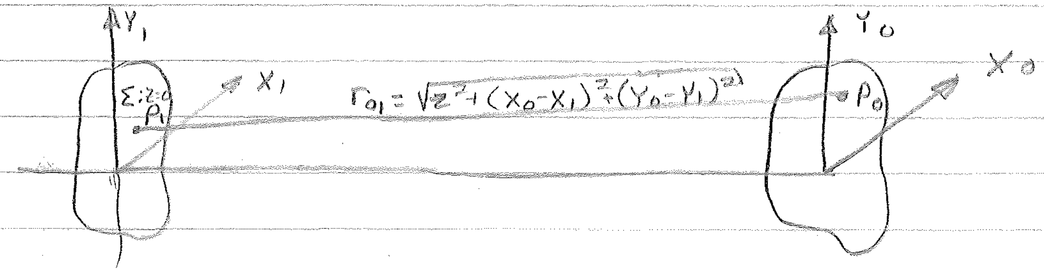
$$=\frac{1}{2\pi}\,\frac{d}{(d^2+r^2)}\,e^{\left[jk(d^2+r^2)^{\frac{1}{2}}\right]}\left[\frac{1}{\sqrt{d^2+r^2}}-jk\right]$$

$$=\frac{-j}{\lambda}\,\frac{d}{d^2+r^2}\,e^{jk(d^2+r^2)^{\frac{1}{2}}}$$

TA TA TA!

3-31-72 (FRI) (HANDOUT DUE MONDAY) DO 4-2, 4-4 (DUE WED) + DISCUSS!





$$U(P_0) = \frac{1}{j\lambda}\int_\Sigma \frac{1}{r_0} e^{jkr_{01}}\, \bar{a}_n \cdot \bar{a}_{r_{01}}\, U(P_1)\, ds$$

LETTING $U(P_1) = 0$ @ P_1 AND Σ

$$U(P_0) = \frac{1}{j\lambda}\int_{-\infty}^{\infty}\!\!\int \frac{1}{r_{01}} e^{jkr_{01}}\, \bar{a}_n \cdot \bar{a}_{r_{01}}\, U(X_1, Y_1)\, dX_1\, dY_1$$

FOR $Z >>$ MAXIMUM DIMENSION OF Σ, $\bar{a}_n \cdot \bar{a}_{r_{01}} = 1$

$r_{01} \sim Z$

BUT DO NOT ASSUME $e^{jkr_{01}} = e^{jkz}$

EXPANSION OF r_{01}; $r_0 = \sqrt{Z^2 + (X_0 - X_1)^2 + (Y_0 - Y_1)^2}$

$$\approx Z\left[1 + \frac{1}{2}\left(\frac{X_0 - X_1}{Z}\right)^2 + \frac{1}{2}\left(\frac{Y_0 - Y_1}{Z}\right)^2\right]$$

$$= Z\left[1 + \frac{1}{2}\left(\frac{X_0}{Z}\right)^2 + \frac{1}{2}\left(\frac{Y_0}{Z}\right)^2 - \left(\frac{X_0X_1 + Y_0Y_1}{Z^2}\right) + \frac{1}{2}\left(\left(\frac{X_1}{Z}\right)^2 + \left(\frac{Y_1}{Z}\right)^2\right)\right]$$

$$\therefore U(P_0) = \frac{1}{j\lambda}\int_{-\infty}^{\infty}\!\!\int \frac{1}{r_{01}} e^{jkr_{01}}\, U(X_1, Y_1)\, dX_1\, dY_1$$

$$= \frac{1}{j\lambda Z}\int_{-\infty}^{\infty}\!\!\int e^{jkz}\, e^{\frac{jk}{2}\left[\left(\frac{X_0}{Z}\right)^2 + \left(\frac{Y_0}{Z}\right)^2\right] Z}\, e^{\frac{jkZ}{2}\left[\left(\frac{X_1}{Z}\right)^2 + \left(\frac{Y_1}{Z}\right)^2\right]}\, e^{-\frac{jkZ}{Z^2}(X_0X_1 + Y_0Y_1)}$$

YIELDING FRESNEL APPROXIMATION

$$U(P_0) = \frac{\exp(jkz)}{j\lambda Z} \cdot \exp\left[\frac{jk}{2Z}(X_0^2 + Y_0^2)\right]$$

$$\cdot \int_{-\infty}^{\infty}\!\!\int \exp\left[\frac{jk}{2Z}(X_1^2 + Y_1^2)\right] U(X_1, Y_1)$$

$$\exp\left(-\frac{jk}{Z}(X_0X_1 + Y_0Y_1)\right] dX_1\, dY_1$$

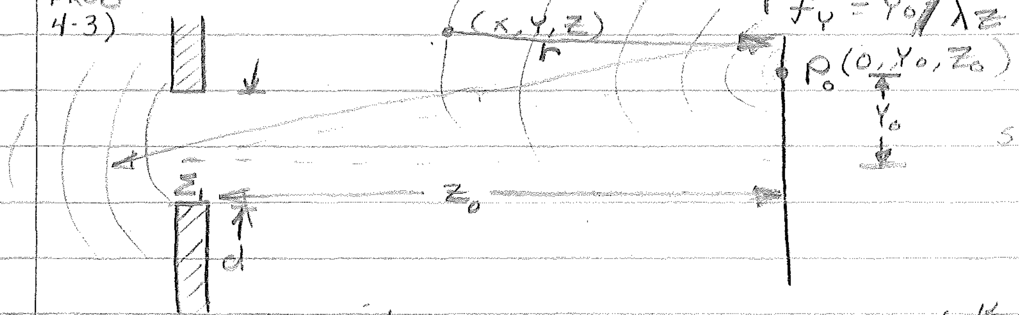
FRAUNHOFER DISCARDS QUADRATIC TERM $e^{\frac{jk}{2z}(x_1^2+y_1^2)}$

$$U(P_0) = \frac{\exp(jkz)}{j\lambda z} e^{\frac{jk}{2z}(x_0^2+y_0^2)} \int\int_{\infty}^{\infty} U(x,y)\, e^{-\frac{jk}{z}(x_0x_1+y_0y_1)}\, dx_1\, dy_1$$

FRESNEL $\alpha\ \mathcal{F}\left\{ \exp \frac{jk}{2z}(x_1^2+y_1^2)\, U(x_1,y_1) \right\} \Big|$ $f_x = \frac{kx_0}{2\pi z} = \frac{x_0}{\lambda z}$, $f_y = \frac{ky_0}{2\pi z} = \frac{y_0}{\lambda z}$

FRAUNHOFER $\alpha\ \mathcal{F}\{U(x_1,y_1)\} \Big|$ $f_x = x_0/\lambda z$, $f_y = y_0/\lambda z$

PROB 4-3)



SHERICAL WAVE e^{jkr}/r

$$\int_{-\infty}^{\infty}\int \exp\left(\frac{jk}{2z}(x_1^2+y_1^2)\right) U(x_1,y_1)\, e^{j\frac{k}{z}(x_0x_1+y_0y_1)}\, dx_1\, dy_1$$

$\therefore$ r = DISTANCE FROM P TO ARBITRARY PT. IN SPACE

$$\Rightarrow r = \sqrt{x^2+(y-y_0)^2+(z-z_0)^2}$$

QUADRATIC APPROX IN PLANE OF APETURE; Z = 0

$$r = \sqrt{z_0^2+x^2+(y-y_0)^2}$$
$$= z_0\sqrt{1+\left(\frac{x}{z_0}\right)^2+\left(\frac{y-y_0}{z}\right)^2}$$
$$\approx z_0\left[1+\frac{1}{2}\left(\frac{x}{z_0}\right)^2+\frac{1}{2}\left(\frac{y-y_0}{z_0}\right)^2\right]$$; GOOD IF $d \ll r$

MAKING PLANE WAVE APPOXIMATION OF SPHERE @ Σ_1

$$\therefore \frac{1}{r}e^{-jkr} \approx \frac{1}{z_0}e^{-jkz_0}\exp\left(\frac{-jk}{2z_0}(x^2+(y-y_0)^2)\right)$$

$$\therefore U(x_1,y_1) \approx \frac{1}{z_0}e^{-jkz_0}\exp\left[\frac{-jk}{2z_0}(x_1^2+(y_1-y_0)^2)\right]$$

AND FOR FRESNEL

$$U(P_0) = \frac{1}{j\lambda z_0}e^{jkz_0}\exp\left[\frac{jk}{2z_0}(x_0^2+y_0^2)\right]$$
$$\int_\Sigma\int \exp\left[\frac{jk}{2z_0}(x_1^2+y_1^2)\right]\left[\frac{1}{z_0}e^{-jkz_0}\right]$$
$$\exp\frac{-jk}{z_0}(x_0x_1+y_0y_1)\, dx_1\, dy_1$$
$$= \frac{1}{j\lambda z_0^2}\exp\left[\frac{jk}{2z_0}(x_0^2+y_0^2-Y_0^2)\right]$$
$$\int_\Sigma\int \exp\left[-j\frac{k}{z_0}(x_0x_1+y_1(y_0-Y_0))\right]$$
$$dx_1\, dy_1$$

INTENSITY $\alpha\ (\mathcal{F}\{\ \})^2 \Big|$ $f_y = y_0 - Y_0/\lambda z_0$, $f_x = x_0/\lambda z_0$

4-1-72 (MON) ; 4-2, 4-6 - DUE WEDNESDAY w 4-4

$e^{i(kz-\omega t)}$; PLANE WAVE

FOURIER-BESSEL TRANSFORMS

$$G(f_X, f_Y) = \int_{-\infty}^{\infty}\int g(x,y)\, e^{-j2\pi(f_X X + f_Y Y)}\,dx\,dy$$

Let $x = r\cos\theta$; $Y = r\sin\theta$

$f_X = \rho\cos\varphi$; $f_Y = \rho\sin\varphi$

$\therefore g(x,y) = g(r\cos\theta, r\sin\theta) = g_R(r)$ FOR A CIRCULARY SYMMETRIC FUNCTION $\Rightarrow$ AXIAL SYMMYTRY)

$$\Rightarrow G(f_X, f_Y) = \int_0^{\infty} g_R(r)\,r\,dr \int_0^{2\theta} d\theta\, e^{-j2\pi r\rho(\cos\theta\cos\varphi + \sin\theta\sin\varphi)}$$

$$= \int_0^{\infty} g_R(r)\,dr\, \underbrace{\int_0^{2\pi} d\theta\, e^{-j2\pi r\rho\cos(\theta-\varphi)}}_{2\pi J_0(2\pi r\rho)}$$

$2\pi J_0(2\pi r\rho)$ = BESSEL FUNCTION OF FIRST KIND, ORDER ZERO (INDEPENDENT OF φ)

$$\Rightarrow G(f_X, f_Y) = 2\pi \int_0^{\infty} g_R(r)\,J_0(2\pi r\rho)\,r\,dr = G_0(\rho) = \mathcal{B}\{g_R(r)\}$$

= FOURIER-BESSEL TRANSFORM

EXAMPLE :

$\mathrm{circ}(r) = 1 \qquad r \le 1$

$\qquad = 0 \qquad$ OTHERWISE

$g_R(r) = \mathrm{circ}(r)$

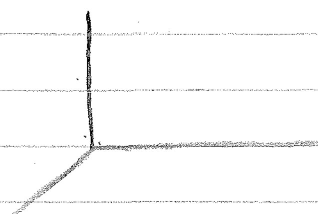
$$\Rightarrow \mathcal{B}\{\mathrm{circ}(r)\} = 2\pi\int_0^1 J_0(2\pi r\rho)\,r\,dr$$

$$= \frac{J_1(2\pi\rho)}{\rho}$$

$\ni J_1(2\pi\rho)$ IS BESSEL FUNCTION OF FIRST KIND, ORDER 1

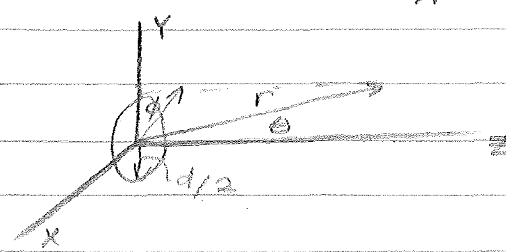
3-4)

$$A_t\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \mathcal{F}\left\{ circ\left(\frac{r}{d/2}\right)\right\}$$
$$= \mathcal{B}\left\{ circ\left(\frac{r}{d/2}\right)\right\}$$
$$= (d/2)^2 \cdot J_1\left(2\pi \frac{d}{2}\rho\right) / d\rho/2$$
$$= \frac{d}{2} \; \frac{J_1\left(\pi \frac{d}{\lambda}(\alpha^2+\beta^2)^{\frac{1}{2}}\right)}{\frac{1}{\lambda}(\alpha^2+\beta^2)^{1/2}}$$
$$= \left(\frac{d}{2}\right)^2 \; \frac{J_1\left(\pi \frac{d}{\lambda}(\alpha^2+\beta^2)\right)}{\frac{d}{\lambda}(\alpha^2+\beta^2)^{1/2}}$$



4-4-72 (WED)

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \left(\frac{d^2}{2}\right) \; \frac{J_1\left(\frac{\pi d}{\lambda}\sin\theta\right)}{\frac{d}{\lambda}\sin\theta}$$



ϕ & θ; SPERICAL CO-ORDINATES

$\alpha = \sin\theta\cos\phi$; $X = r\sin\theta\cos\phi$

$\beta = \sin\theta\sin\phi$; $Y = r\sin\phi\cos\theta$

α & β ARE DIRECTION COSINES RELATIVE TO THE X & Y AXES, RESPECTIVELY (SEE FIG 2-3, PG 16)

NOTE THAT THE LARGER $\frac{d}{\lambda}\sin\theta$, THE SMALLER A_t

$$4\text{-}6)\ \psi = \int_{-\infty}^{\infty}\!\!\int e^{j\frac{k}{2\pi}(x_1^2+y_1^2)}\, e^{-j2\pi(f_x x + f_y y)}\, dx_1 dy_1 \Leftarrow \text{SEPARABLE}$$

$$\int_{-\infty}^{\infty} e^{j\frac{k}{2\pi}x_1^2}\, e^{-j2\pi f_x x}\, dx_1 = \int_{-\infty}^{\infty} e^{j\frac{k}{2\pi}\underbrace{(x_1^2 - \frac{4\pi^2}{k} f_x x_1)}_{\text{COMPLETE SQUARE}}}\, dx_1$$

$$x_1^2 - bx + c = (x-\alpha)^2 = x_1^2 - 2\alpha x_1 + \alpha^2 \Rightarrow \alpha = b/2;\ \alpha^2 = \left(\frac{b}{2}\right)^2$$

$$b = \frac{(2\pi)^2}{k} f_x \Rightarrow (b/2)^2 = \frac{4\pi^2}{k^2} f_x^2 = \frac{4\pi^4}{k^2} f_x^2$$

$$\Rightarrow \psi = \int e^{-j\frac{k}{2\pi}\left(x_1^2 - 4\pi^2 f_x x_1/k + \frac{4\pi^4}{k^2} f_x^2 - \frac{4\pi^4}{k^2} f_x^2\right)}\, dx_1$$

$$= \int_{-\infty}^{\infty} \exp\left[-j\frac{2\pi^3}{k} f_x^2\right] \exp\left[\frac{jk}{2\pi}\left(x_1 - \frac{2\pi^2}{k} f_x\right)^2\right] dx_1$$

LET $\eta = \sqrt{k/2\pi}\left(x_1 - \frac{2\pi^2}{k} f_x\right)$

$$\eta^2 = \frac{k}{2\pi}\left(x_1 - \frac{2\pi^2}{k} f_x\right)^2$$

$$d\eta = \sqrt{k/2\pi}\, dx_1$$

$$\psi = \sqrt{\frac{2\pi}{k}}\, \exp\left(-j\frac{2\pi^3}{k} f_x^2\right) \int_{-\infty}^{\infty} e^{ju^2}\, du$$

$$\int_{-\infty}^{\infty} e^{ju^2}\, du = \sqrt{\frac{\pi}{2}}\,(1+j)$$

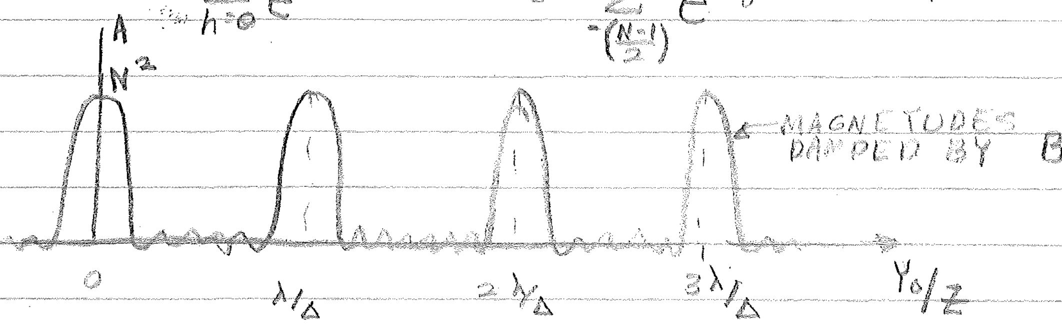
$$\Rightarrow \psi = \pi\sqrt{\frac{2}{k}}\, e^{-j\left(\frac{2\pi^3}{k} f_x^2 - \pi/4\right)}$$

4-2) ANS)

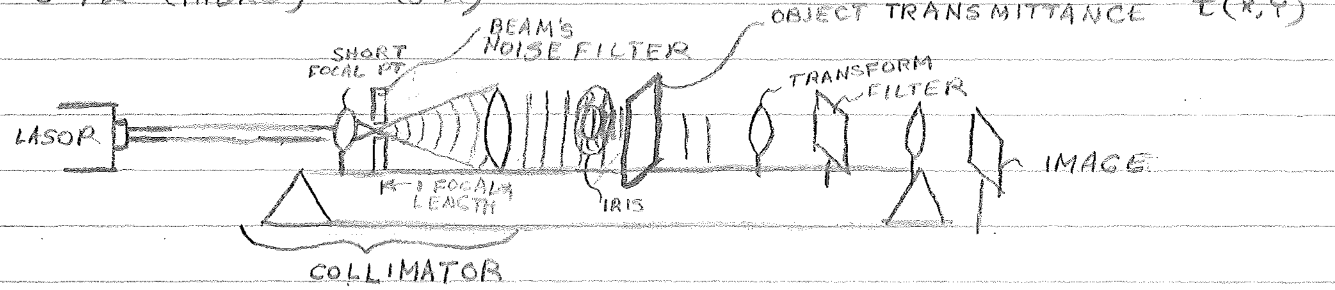
$$I(x_0,y_0) = \left(\frac{XY}{\lambda Z}\right)^2 \operatorname{sinc}^2\left(\frac{x_0 X}{\lambda Z}\right) \operatorname{sinc}^2\left(\frac{y_0 Y}{\lambda Z}\right) \left(\frac{\sin N\frac{\pi}{\lambda}\Delta y_0/z}{\sin \frac{\pi}{\lambda}\Delta y_0/z}\right)^2$$

$$U(x_0,y_0) = \frac{XY}{\lambda Z} \operatorname{sinc}\left(\frac{x_0 X}{\lambda Z}\right) \underbrace{\operatorname{sinc}\left(\frac{y_0 Y}{\lambda Z}\right)}_{B} \underbrace{\left(\frac{1-\exp(-j2\pi N\Delta y_0/(\lambda Z))}{1-\exp(-j2\pi\Delta y_0/(\lambda Z))}\right)}_{A}$$

$$A = e^{j2\pi(N-1)f_Y} \sum_{n=0}^{N-1} e^{-j2\pi n\Delta f_Y} = \sum_{-(\frac{N-1}{2})}^{(\frac{N-1}{2})} e^{-j2\pi n\Delta f_Y}$$

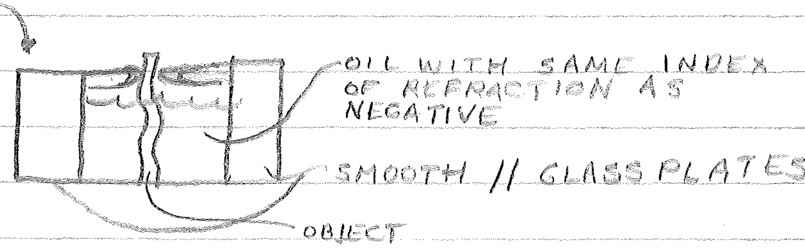


4-6-72 (THURS) (5-2)



LIQUID GATE

OBJECT NOT FLAT
⇒ PHASE MODULATION (BAD)

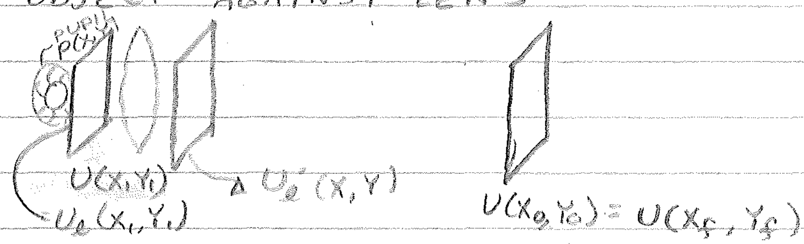


4-20-72 (5-2 DUE FRIDAY; LOOK @ 5-8)

$$U(x_0,y_0) = \frac{\exp(jkz)}{j\lambda z}\int_{-\infty}^{\infty}\int U(x_1,y_1)\exp\left\{\frac{jk}{2z}\left[(x_0-x_1)^2+(y_0-y_1)^2\right]\right\}dx_1\,dy_1$$

$$= \frac{\exp(jkz)}{j\lambda z}\exp\left[\frac{jk}{2z}(x_0^2+y_0^2)\right]\int_{\infty}^{\infty}\int U(x_1,y_1)\exp\left[\frac{jk}{2z}(x_1^2+y_1^2)\right]$$
$$\exp\left[\frac{-jk}{z}(x_0x_1+y_0y_1)\right]dx_1\,dy_1$$

① OBJECT AGAINST LENS



$$U_l'(x,y) = U_l(x,y)\underbrace{t_o(x,y)}_{\text{OBJECT FUNCTION}}\,e^{\frac{-jk}{2f}(x^2+y^2)}$$

LET $U_l(x,y) = A$

$$U_l'(x,y) = A\,t_o(x,y)\,e^{\frac{-jk}{2f}(x^2+y^2)}\,P(x,y)$$

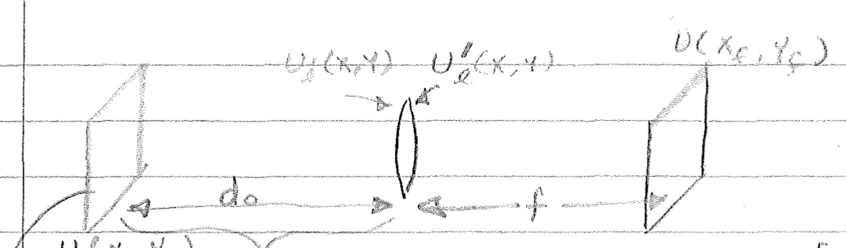
NEGECTING e^{jkz}

$$U(x_f,y_f) = \frac{1}{j\lambda z}\exp\left[\frac{jk}{2z}(x_0^2+y_0^2)\right]\int_{\infty}^{\infty}\int A\,t_o(x,y)$$
$$\exp\left[\frac{-jk}{2f}(x^2+y^2)\right]P(x,y)$$
$$\exp\left[\frac{-jk}{z}(xx_f+yy_f)\right]\exp\left[\frac{jk}{2z}(x^2+y^2)\right]dx\,dy$$

FOR $z = f$ (FOCAL DISTANCE)

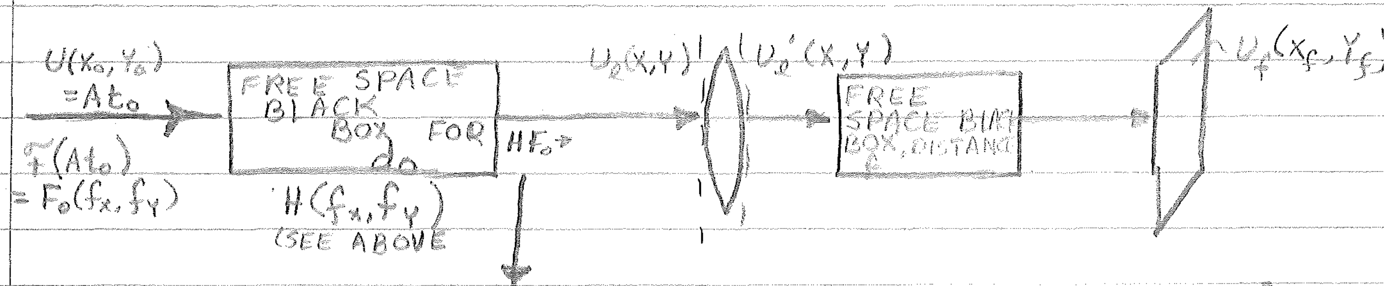
$$U(x_f,y_f) = \frac{1}{j\lambda z}\mathcal{F}\{A\,t_o(x,y)\,P(x,y)\}\exp\left[\frac{jk}{2f}(x_f^2+y_f^2)\right]$$

WHERE $f_x = \frac{x}{\lambda f}$ AND $f_y = \frac{y}{\lambda f}$



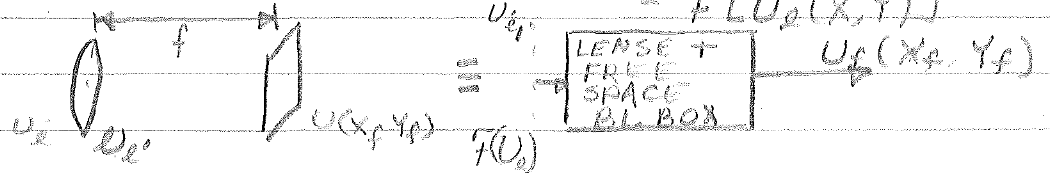
$$H(f_x, f_y) = \exp(jkd_o)\exp[-j\pi\lambda d_o(f_x^2 + f_y^2)]$$

= FREE SPACE TRANSFER FUNCTION



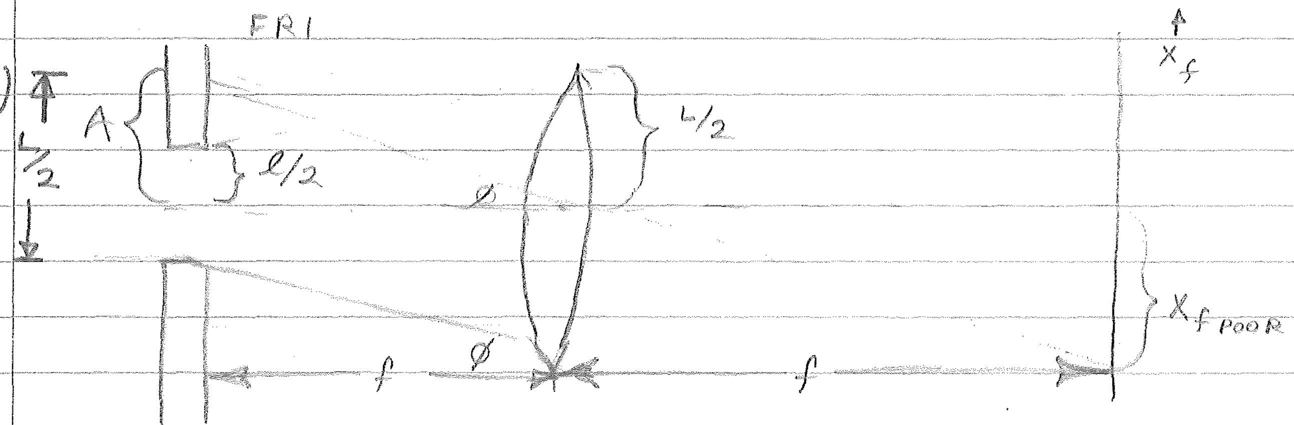
$$= F_o(f_x, f_y)\exp[-j2\pi\lambda d_o(f_x^2 + f_y^2)]$$
$$= \mathcal{F}[U_l(x,y)]$$

BUT,



$$= \frac{1}{j\lambda z}\exp\left[\frac{jk}{2f}(x_f^2 + y_f^2)\right]\mathcal{F}\{U_l\}$$
$$\ni \mathcal{F}(U_l) = \exp[-j\pi\lambda d_o(f_x^2 + f_y^2)]\ \mathcal{F}\{U_o\}$$
FOR $z_o = f$; $U_f(x_f, y_f) = \mathcal{F}\{U_o\}\exp[-j\pi\lambda d_o(f_x^2 + f_y^2)]$

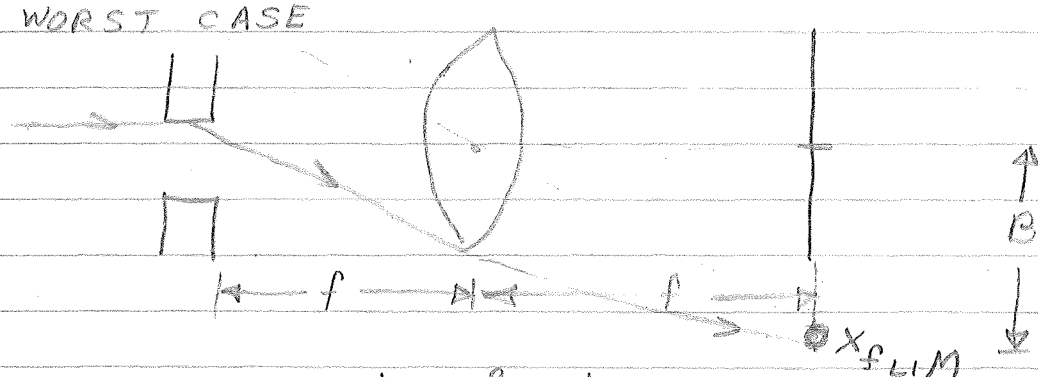
5-4)



$|A| = |X_{f\,POOR}|$

$\frac{L}{2} = A + \frac{l}{2} \Longrightarrow A = \frac{L}{2} - \frac{l}{2} = \frac{1}{2}(L - l) = X_f$

WORST CASE



$B = X_{f\,LIM} = \frac{L}{2} + \frac{l}{2} = \frac{1}{2}(L + l)$

GOOD VALUES:

$L = 4\,cm \qquad f = 50\,cm$

$l = 2\,cm \qquad \lambda = 6 \times 10^{-7}\,m$

$\Rightarrow f_{POOR} = \frac{X_f}{\lambda f} = \frac{1}{2}(.04 - .02)\left(\frac{1}{6 \times 10^{-7}}\right)/.5 \quad \left(\frac{1}{m}\right)$

$= \frac{1}{3} \times 10^5\,m = \frac{1}{3} \times 10^2/mm$

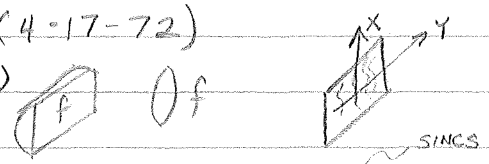
$f_{lim} = 100$ cycles/mm

5-8 DUE MONDAY

FINISH CHAPTER

MON (4-17-72)

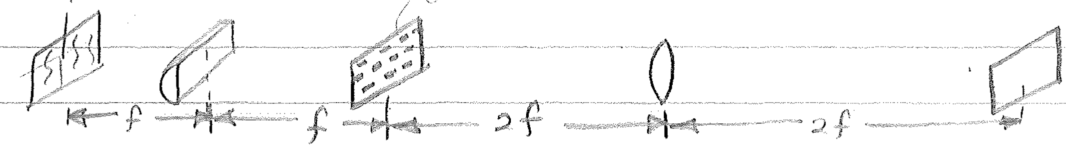
5-6) b)



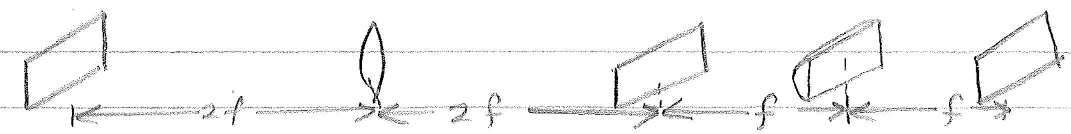
$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{2}{f} \quad \text{(TRANSFORM)}$$
$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \quad \text{(IMAGING)}$$

SINCS

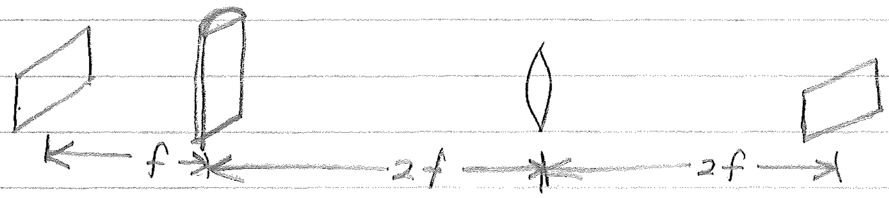
FOR UNIT MAGNIFICATION $d_o = d_i$



OR

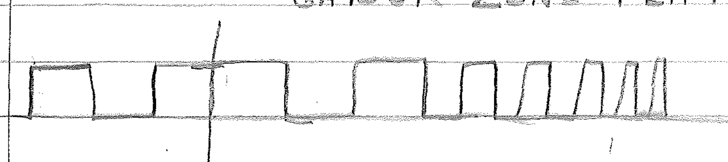


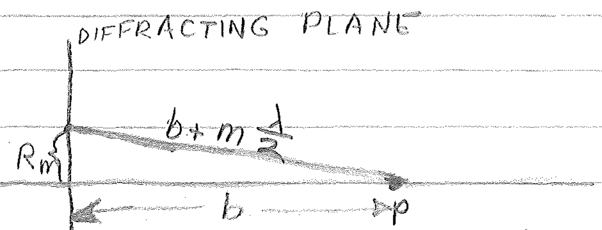
OR



(5-5 & 5-10)

$$t(r) = \underbrace{\left(\tfrac{1}{2} + \tfrac{1}{2}\cos \alpha r^2\right)}_{\text{GABOR ZONE PLATE}} \mathrm{circ}\left(\frac{r}{l}\right)$$



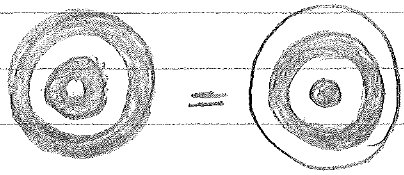


$$\Rightarrow R_m^2 = \left(b + \frac{m\lambda}{2}\right)^2 - b^2 = bm\lambda + \left(\frac{m\lambda}{2}\right)^2$$
$$\simeq bm\lambda \Rightarrow R_m = \sqrt{bm\lambda} = \sqrt{b\lambda}\,\sqrt{m}$$

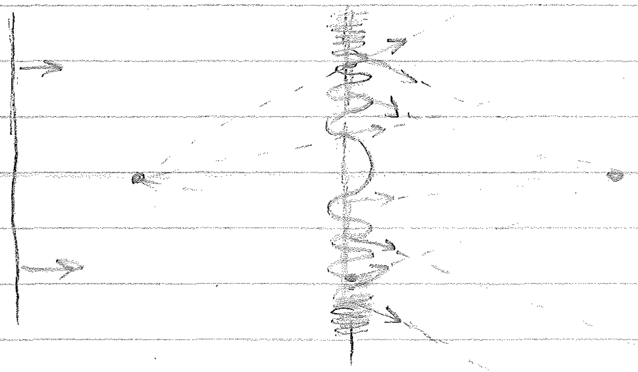
IF $\cos \alpha r^2 = 0 \Rightarrow r = \sqrt{\frac{\pi}{2\alpha}}\,\sqrt{m}$ ← NOTE CORRELATION

NOW focal LENGTH $= b = R_m^2/m\lambda = (R\sqrt{m})^2/m\lambda = R_1^2/\lambda \quad \left(f \propto \frac{1}{\lambda}\right)$

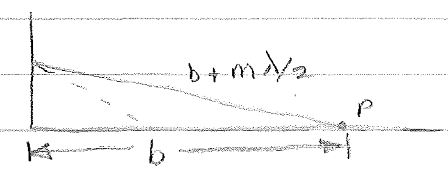
THUS



YIELD LIGHT PT. @ P FOR BOTH CONFIGURATIONS. (POISSON'S SPOT)



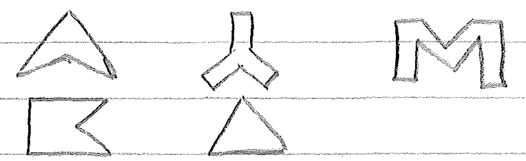
FRESNEL ZONE PLATES GOOD FOR "LENSES"

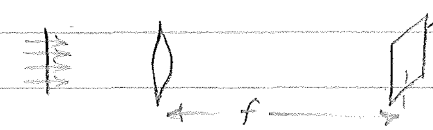


FOCAL PTS @ $b, \frac{1}{3}b, \frac{1}{5}b, \ldots$, EACH DECREASING FOCAL PT HAS LESS ENERGY

4-18-71 (WED) 6-3 DUE MONDAY

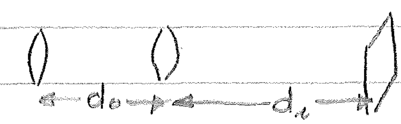
LAB APERTURES



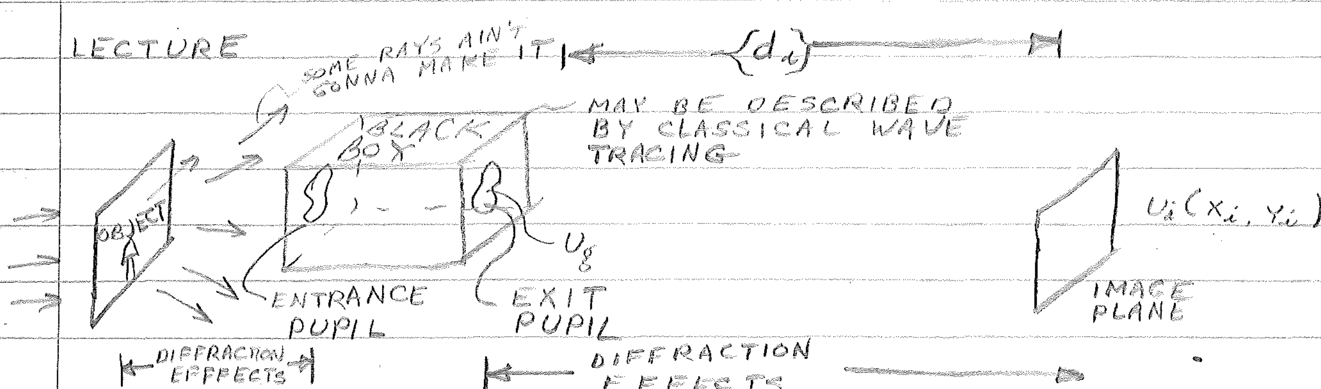


~MUST SEE TRANSFORM
1) MAKE IMAGE SMALL
2) USE MICROSCOPE
3) MAGNIFIER LENSE

$$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} \; ; M = d_i / d_o$$



LECTURE



NEED ONLY CONSIDER SMALLEST PUPIL

ENTRANCE PUPIL MAY BE LOOKED UPON AS LOW PASS FILTER

$$U_i(x_i, y_i) = \iint h(x_i - \tilde{x}_o, y_i - \tilde{y}_o) U_g(\tilde{x}_o \tilde{y}_o) \, d\tilde{x}_o \, d\tilde{y}_o$$

EXIT PUPIL

$$\tilde{x}_o = M x_o \; ; \; \tilde{x} = x/\lambda d_i \; ; \; h(x_i, y_i) = \int_{-\infty}^{\infty}\!\!\int P(\lambda d_i \tilde{x}, \lambda d_i \tilde{y}) \exp\left(-j 2\pi \{x_i \tilde{x} + y_i \tilde{y}\}\right) d\tilde{x} \, d\tilde{y}$$

$$U_o \longrightarrow \boxed{\begin{array}{l} h(x,y) \\ = \delta(x,y) \\ \text{IDEALLY} \end{array}} \longrightarrow U_i \quad \text{, BUT } h(x,y) = P(\lambda d_i \tilde{x}, \lambda d_i \tilde{y})$$

$$U_i = U_i(\omega) * U_g$$

LET $G_g(f_x, f_y) = \mathcal{F}[U_g(\tilde{x}_0, \tilde{y}_0)]$

$G_i(f_x, f_y) = \mathcal{F}[U_i(x_i, y_i)]$

$H(f_x, f_y) = \mathcal{F}[\tilde{h}(x_i, y_i)]$

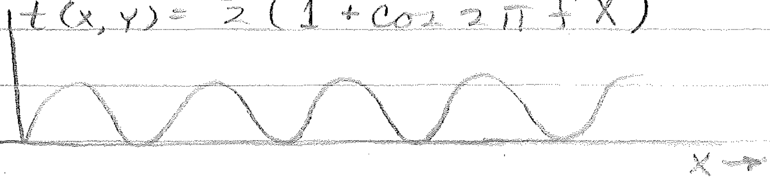
$G_i = H \cdot G_g$, BUT $H(f_x, f_y) = \mathcal{F}[\tilde{h}]$

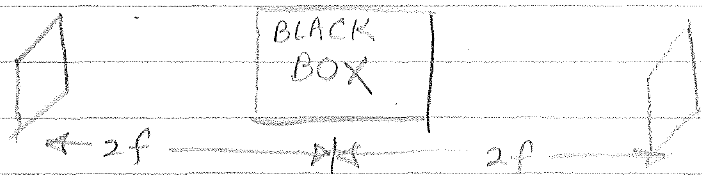
$= \mathcal{F}[\mathcal{F}\{P(x, y)\}]$

$= P(-\lambda d_i f_x, -\lambda d_i f_y)$

4-24-72 (MON)

6-4) $t(x, y) = \frac{1}{2}(1 + \cos 2\pi \tilde{f} x)$ (AMPLITUDE)





$T(x, y) = \frac{1}{2}\delta(f_x) + \frac{1}{4}\delta(f_x - \tilde{f}) + \frac{1}{4}(f_x + \tilde{f})$

$H(f_x, f_y) = \cancel{P[\lambda d_i f_x, \lambda d_i f_y]}\, \exp[jkW(\lambda d_i f_x, \lambda d_i f_y)]$

ASSUME = 1

PHASE PHACTOR DUE TO ABERATIONS

IF $\frac{1}{d_o} + \frac{1}{d_i} - \frac{1}{f} = \epsilon$

$\Rightarrow W(\lambda d_i f_x, \lambda d_i f_y) = \frac{\epsilon[(\lambda d_i f_x)^2 + (\lambda d_i f_y)^2]}{2}$

$U_i(f_x, f_y) = T \cdot H$

$= [\frac{1}{2}\delta(f_x) + \frac{1}{4}\delta(f_x - \tilde{f}) + \frac{1}{4}\delta(f_x + \tilde{f})]$

$\exp[jk\epsilon[(\lambda d_i f_x)^2 + (\lambda d_i f_y)]/2]$

$= \frac{1}{2}\delta(f_x) + \frac{1}{4}\delta(f_x - \tilde{f})e^{jk\epsilon(\lambda d_i \tilde{f})^2/2}$

$+ \frac{1}{4}\delta(f_x + \tilde{f})e^{jk\epsilon(\lambda d_i \tilde{f})^2/2}$

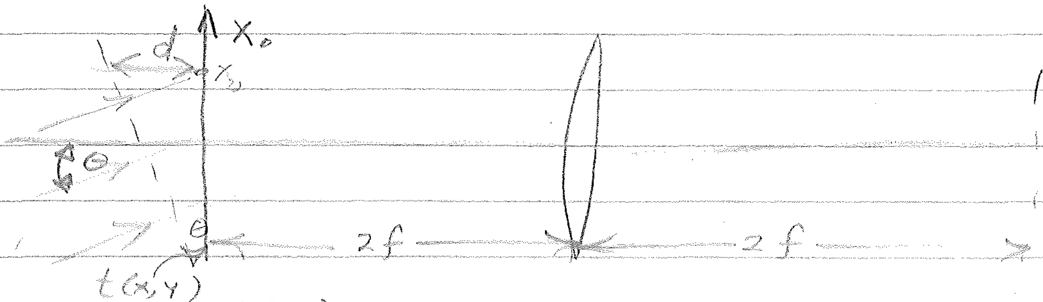
$= \frac{1}{2}\delta(f_x) + \frac{1}{4}[\delta(f_x + \tilde{f}) + \delta(f_x - \tilde{f})]\exp[jk\epsilon(\lambda d_i \tilde{f})^2]$

NOW

$\exp[k\epsilon(\lambda d_i \tilde{f})^2/2] = 1$ WHEN ARGUMENT $= 2n\pi$,

WHERE OBJECT WILL BE REGAINED.

(6-5) $\mathcal{F}\{e^{j\omega t}\} = \delta(\omega-\omega_0)$ (TILTED PLANE WAVE)



$U_i = Ae^{j\phi(x_0)} = A\exp(jkx_0)\tan\theta \Rightarrow \theta = \tan^{-1}\frac{d}{x_0}$

$t(x,y) = \frac{1}{2}[1+\cos 2\pi\tilde{f}x]$

$\Rightarrow U_g(x_0) = \frac{A}{2}[1+\cos 2\pi\tilde{f}x_0]\underbrace{\exp[jkx_0\tan\theta]}_{j2\pi f_s x_0}$

$\Rightarrow f_s = \tan\theta/\lambda$

$\Rightarrow \mathcal{F}\{U_g\} = \frac{A}{2}\delta(f_x) + \frac{A}{4}\delta(f_x-\tilde{f}) + \frac{A}{4}\delta(f_x+\tilde{f})$

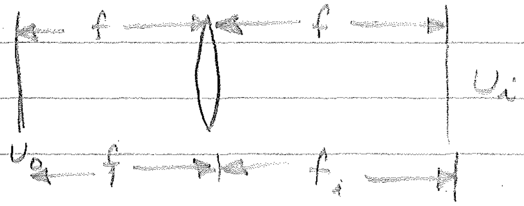
SHIFTED BY $f_s = \tan\theta/\lambda$ IN THE POSITIVE $+x_0$ DIRECTION

NOW $H = \mathrm{circ}\left(\sqrt{f_x^2+f_y^2}\Big/\frac{\ell}{2\lambda d_i}\right)$

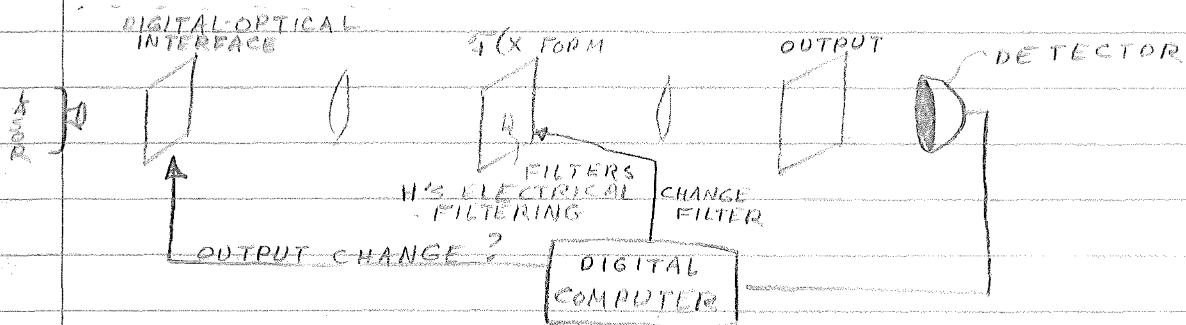
MAX DIST: $F_s = \ell/2\lambda d_i = \tan\theta$

$\Rightarrow \tan\theta < \lambda/$

4-26-72 (WED)



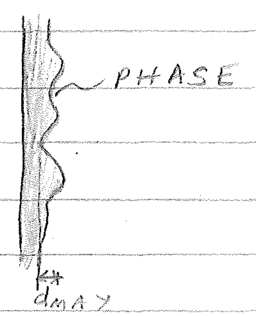
$H = e^{-jkw_i^2}$ GIVES H WHICH GIVES $\mathcal{F}$ X FROM X PHASE FACTOR



PHASE MODULATION

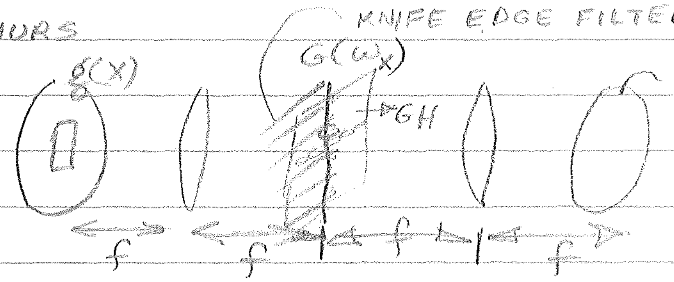
$t(x,y) = e^{j\phi(x,y)} \simeq 1 + j\phi(x,y)$ FOR SMALL $\phi(x,y)$

($\phi \sim 4$ RAD)



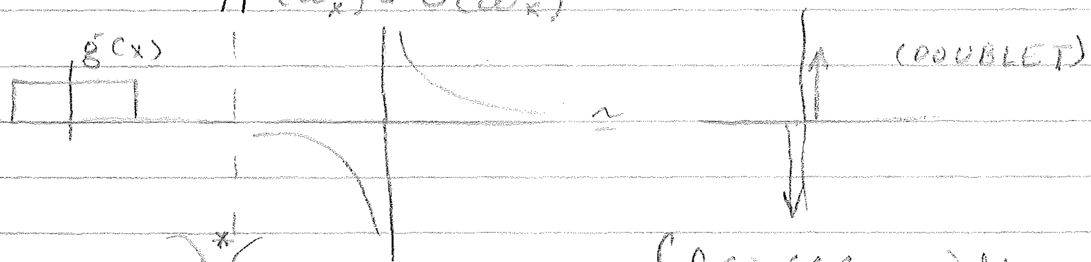
$\Rightarrow \phi_{MAX} = \frac{2\pi(N-1)}{\lambda} \cdot d_{MAX}$

THURS

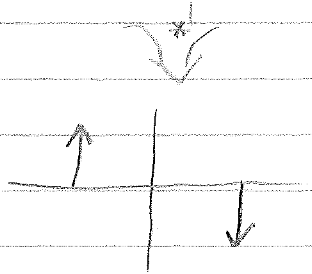


$$\mathcal{F}(G\cdot H) = g(x) * \left(\pi\delta(x) + \frac{1}{j}\frac{1}{x}\right)$$
$$= \pi g(x) - j\left(g(x) * \frac{1}{x}\right)$$

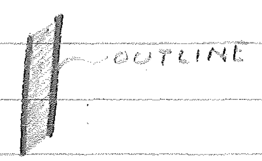
$H(\omega_x) = U(\omega_x)$

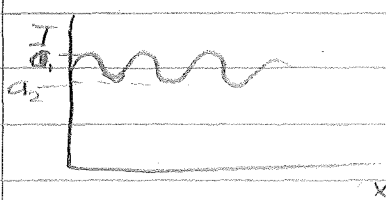


NOW $\int f(x)\,\delta'(t-t_0)\,dt = -f'(t_0)$



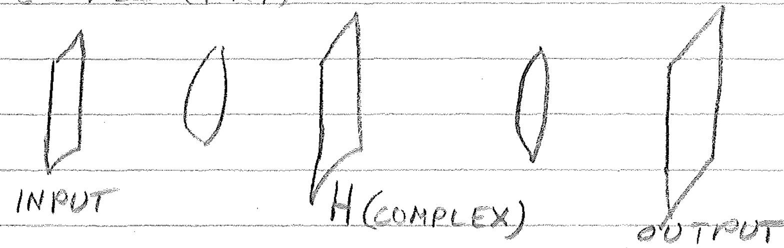
WILL SEE



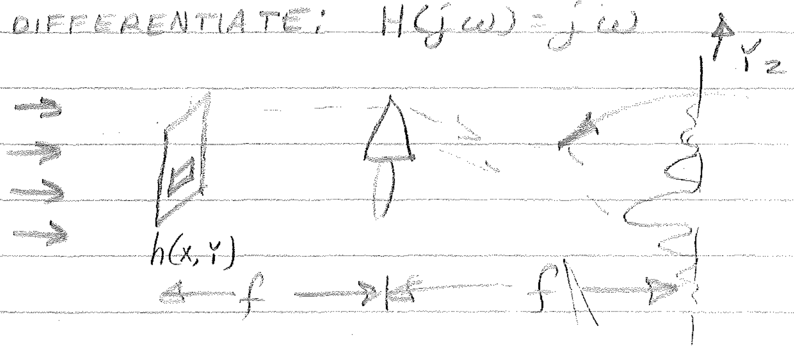


CONTRAST = a_2/a_1

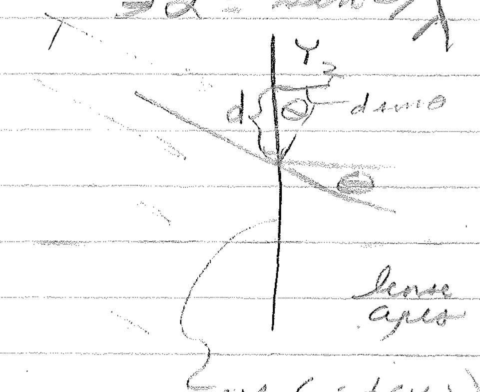
5-5-72 (FRI)



TO DIFFERENTIATE: $H(j\omega) = j\omega$



$U_r = r_0 \exp(-j\, 2\pi \alpha Y_2)$
$\ni \alpha = \sin\theta / \lambda$



$$U_r = \exp(-j\underbrace{2\pi \alpha Y_2}_{\phi(Y_2)}) \ni \alpha = \frac{\sin\theta}{\lambda}$$

$\exp(-j\phi(Y_2))$
$= \exp(-j\, k\, d(Y_2))$

$$I(x_2, Y_2) = \left| \left[r_0 \exp(-j\phi(Y_2)) \right] + \frac{1}{\lambda f} H \right|^2$$

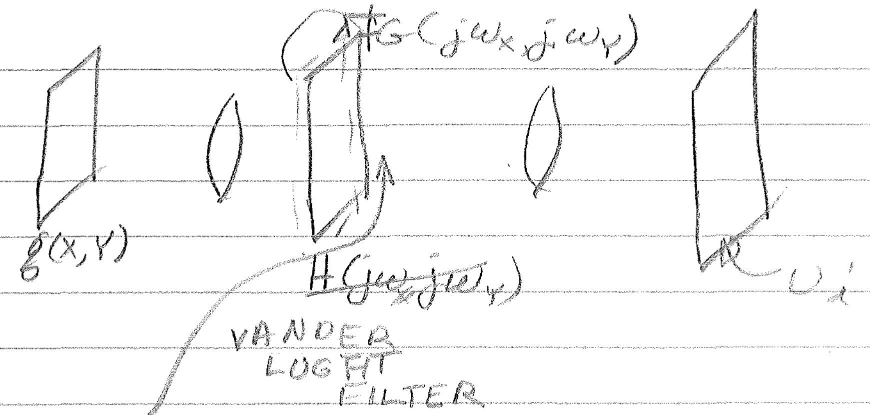
$$= \left(r_0 \exp(-j\phi(Y_2)) + \frac{1}{\lambda f} H \right)(\text{CONJUGATE})$$

$$= r_0^2 + \frac{r_0}{\lambda f} \exp[j\phi(Y_2)] H + \frac{r_0}{\lambda f} e^{j\phi(Y_2)} H^*$$
$$+ \frac{1}{(\lambda f)^2} |H|^2$$

IF $H = A \exp[-j\psi]$

THEN

$$I = r_0^2 + \frac{A^2}{(\lambda f)^2} + \frac{r_0 A}{\lambda f}[\cos\phi + j\sin\phi][\cos\psi - j\sin\psi]$$
$$+ (\cos\phi - j\sin\phi)(\cos\psi + j\sin\psi)$$
$$= r_0^2 + \frac{A^2}{(\lambda f)^2}(\cos\phi\cos\psi + \sin\phi\sin\psi)$$
$$= r_0^2 + \frac{A^2}{(\lambda f)^2}\cos(\phi - \psi)$$

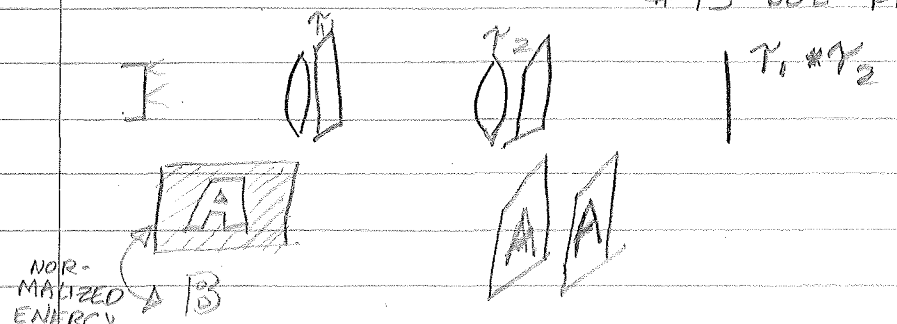


$$\text{GET } \mathcal{F}G\, r_0^2 + \frac{G|H|^2}{(\lambda f)^3} + \frac{r_0}{(\lambda f)^2} G H \exp(j\phi) + \frac{r_0}{(\lambda f)^2} G H^* \exp(-j\phi) \qquad (7\text{-}11)$$

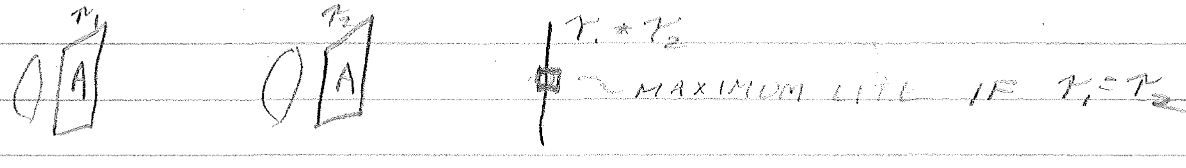
$$U_i \propto g r_0^2 + \frac{1}{(\lambda f)^2} g * h * h^*(-x_i, -y_i) + \frac{r_0}{\lambda f} g * h * \delta(x_i, y_i - f(\phi)) + \underbrace{\frac{r_0}{\lambda f} g * h^*(-x_i, y_i)}_{\text{CROSS CORRELATION}} * \delta(x_i\, y_i - f(\phi))\Big]\Big|$$

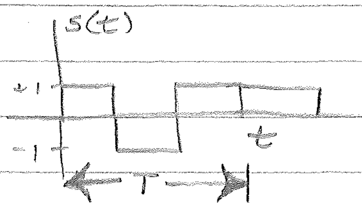
6-8-72 (MON) CHAPT 7, #9 - DUE WEDNESDAY

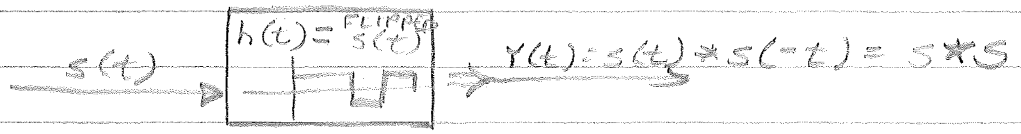
#13 DUE FRIDAY



(TEMPLATE MATCHING)



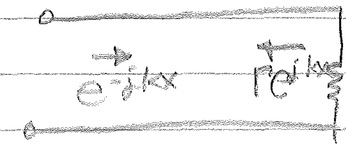




@ $t = T$, $S * S$ WILL OVERLAP, YIELDING MAXIMUM VALUE

NOW $H(\omega) = S(-\omega) = S^*(\omega)$

WED (5-10-72)



$$|A|^2\left[e^{-jkx}+\Gamma e^{jkx}\right]^2$$
$$=A^2\left[e^{-jkx}+\Gamma e^{jkx}\right]\left[e^{jkx}+\Gamma e^{-jkx}\right]$$
$$=A^2\left[1+\Gamma^2+\Gamma\left(e^{jkx}+e^{-jkx}\right)\right]$$
$$=A^2\left[1+\Gamma^2+2\Gamma\cos kx\right]$$

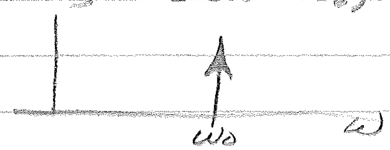
$e^{j\omega_0 t}=\cos\omega_0 t+j\sin\omega_0 t$ (ANALYTIC SIGNAL)

$\cos\omega t$ & $\sin\omega t$ ARE HILBERT TRANSFOR

$V(t)$ = ANALYTIC SIGNAL $= V^r(t)+jV^i(t)$

$\ni V^r(t)$ & $V^i(t)$ ARE HILBERT XFØRMS

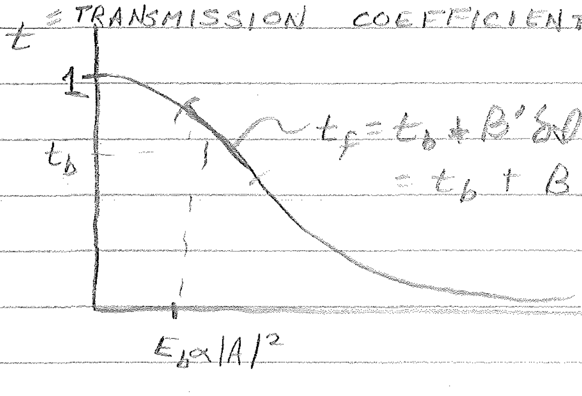
$$\mathcal{F}\{e^{j\omega t}\}=\delta(\omega-\omega_0)$$



5-15-72 (MON)

t = TRANSMISSION COEFFICIENT (NEGATIVE TRANSPARENCY)

SLOPE IS NEGATIVE

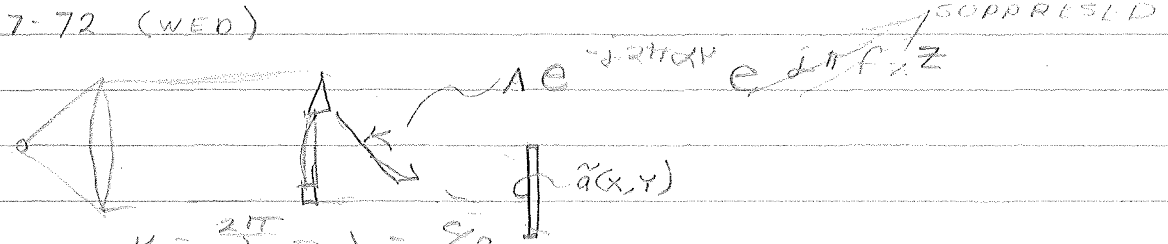


$$t_f = t_b + B'\delta\varnothing$$
$$= t_b + B'\left(|\tilde{a}|^2 + \hat{A}^*\tilde{a} + \hat{A}\tilde{a}^*\right)$$

ASSUME $|\tilde{A}^2|$ IS UNIFORM

EXPOSURE (TIME)

5-17-72 (WED)



$K = \frac{2\pi}{\lambda} \Rightarrow \lambda = c/f$

$k_Y = k\sin\theta = \frac{2\pi}{\lambda}\sin\theta = 2\pi f_Y \Rightarrow f_Y = \alpha = \frac{\sin\theta}{\lambda}$

$U = a e^{j2\pi\alpha Y} + \tilde{a}(x,y) = A e^{j2\pi\alpha Y} + a(x,y) e^{j\phi(x,y)}$

REFERENCE PHASE

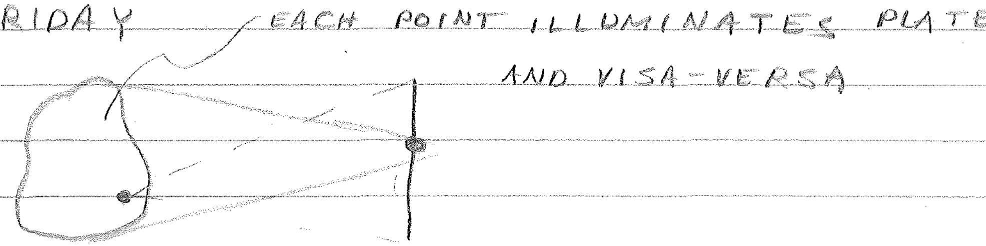
$I = |U|^2 = [A e^{j2\pi\alpha Y} + a e^{j\phi}][A^* e^{-j2\pi\alpha Y} + a^* e^{-j\phi}]$

$\cancel{|U|^2} = A^2 + a^2(x,y) + 2Aa(x,y)\cos[2\pi\alpha Y - \phi(x,y)]$

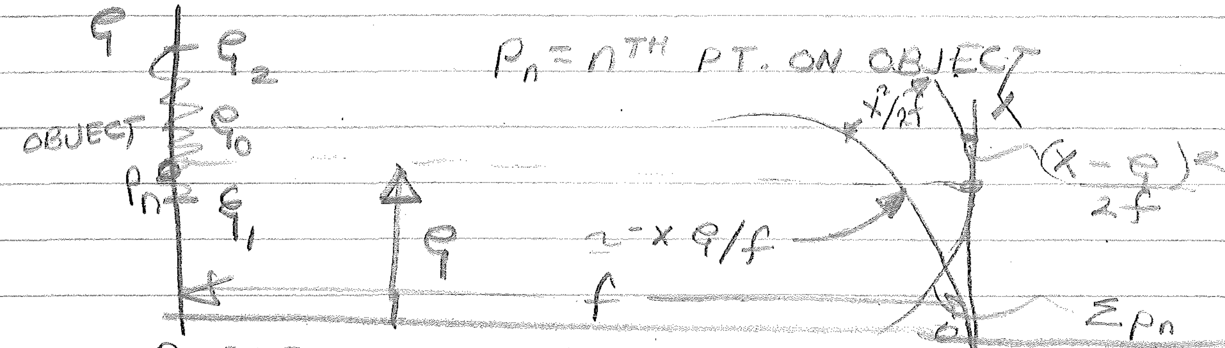
(AMPLITUDE AND PHASE MODULATED)

$\tilde{t}_f(x,y) = \tilde{t}_b + \beta'[|\tilde{a}|^2 + A\tilde{a}e^{j2\pi\alpha Y} + A\tilde{a}^* e^{-j2\pi\alpha Y}]$

FRIDAY



MATH ANALYSIS OF LESLESS F XFORM HOLOG
(G. STOKE, SECT 3.3, p.121



R=REFRENCE PT
(ASSUME 1 DIMENSIONAL OBJECT

TRANSMISSION $T(\rho - \rho_0)$ Σ_R

Σ_{P_n} = CONSTANT PHASE SURFACE FOR P_n
Σ_R = " " " " R

SPHERICAL REFERENCE WAVE

$$U_r = A_0 e^{(j\pi/\lambda f)x^2} \; ; \; A_0 = \text{Real}$$

UNDER ILLUMINATION OF NORMALLY INCIDENT, UNIT AMPLITUDE, PLANE WAVE, OBJECT (POINT SOURCES) PRODUCE SPHERICAL WAVES:

$$dV_0 = t(\rho - \rho_0)\, d\rho \cdot \exp\left(\frac{j\pi}{\lambda f}(x-\rho)^2\right)$$

$$V_0 = \int_{\rho_1}^{\rho_2} t(\rho - \rho_0) \exp\{j\pi/\lambda f\,(x-\rho)^2\}\, d\rho$$

IN FAR FIELD (FOR A SMALL OBJECT & LARGE RECORDING PLATE FOR HI-RESOLUTION)

$$(x-\rho)^2 \approx x^2 - 2x\rho$$

$$\therefore V_0(x) = e^{j(\pi/\lambda f)x^2} \int_{\rho_1}^{\rho_2} t(\rho - \rho_0)\, e^{-j\frac{2\pi}{\lambda f}x\rho}\, d\rho$$

$$= e^{j(\pi/\lambda f)x^2} \int_{-\infty}^{\infty} p(\rho - \rho_0)\, t(\rho - \rho_0)\, e^{-j(2\pi/\lambda f)x\rho}\, d\rho$$

$$= e^{j\left(\frac{\pi}{\lambda f}\right)x^2} \mathcal{F}\left[p(\varphi-\varphi_0)\,t(\varphi-\varphi_0)\right]_{f_x = x/f\lambda}$$

$$= e^{j\frac{\pi}{\lambda f}x^2} e^{-j\varphi_0 2\pi f_x} \mathcal{F}\left[p(\varphi-\varphi_0)\,t(\varphi-\varphi_0)\right]$$

$$= e^{j\frac{\pi}{\lambda f}x^2} e^{-j2\pi x\varphi_0/\lambda f}\, T\left(\frac{x}{\lambda f}\right) \ni T\left(\frac{x}{\lambda f}\right) = \mathcal{F}(p(t))$$

$$\therefore U_f(x) = A_0 e^{j\frac{\pi}{\lambda f}x^2} + e^{j\frac{\pi}{\lambda f}x^2} e^{-j2\pi x\varphi_0/\lambda f}\, T(x/\lambda f)$$

$$I(x) = |U_f(x)|^2 = A_0^2 + |T(x/\lambda f)|^2 + A_0 e^{-j\frac{2\pi x\varphi_0}{\lambda f}}\, T(x/\lambda f)$$

$$+ A_0 e^{j2\pi x\varphi_0/\lambda f}\, T^*(x/\lambda f)$$

$$T^*(x/\lambda f) = \int_{-\infty}^{\infty} t^*(\varphi)\, e^{j(2\pi/\lambda f)x\varphi}\, d\varphi$$

$$= \mathcal{F}^{-1}(t^*(\varphi))$$

FILM TRANSPARENCY TRANSMITTANCE

$$t_f(x) = t_b + \beta'\left\{|T(x/\lambda f)|^2 + A_0 e^{-j2\pi\left(\frac{x\varphi_0}{\lambda f}\right)}\right.$$

$$\left. \times T(x/\lambda f) + A_0 e^{j2\pi(x\varphi_0/\lambda f)}\, T^*(x/\lambda f)\right\}$$

$\Rightarrow$ FOURIER X-FORM HOLOGRAM

RECONSTRUCTION OF HOLOGRAM. TAKE $\mathcal{F}$-XFORM AGAIN. DO THIS BY ~~BY~~ PUTTING THE HOLOG. $t_f(x)$ IN FRONT FOCAL PLANE, f_L, OF A (+) LENS (EQ 5-19)

$$\Rightarrow U_{f_L}(\varphi_L) = \int_{-\infty}^{\infty} t_f(x)\, e^{-j2\pi x/\lambda f_L\, x\varphi_L}\, dx$$

$$= \mathcal{F}\{t_f(x)\}\big|_{f_x = \varphi_L/\lambda f_L}$$

$$U_f = \mathcal{F}\{t_b\}\big|_{f_x = \frac{\varphi_L}{\lambda f_L}} = t_b\, \delta(\varphi_L/\lambda f_L) \quad \text{(D.C)}$$

$$U_{f_{L2}} = \mathcal{F}\{\beta' |t(\varphi)|^2\}\big|_{x/\lambda f} = \beta' \mathcal{F}\{\mathcal{F}(t(\varphi))_{x/\lambda f} \cdot \mathcal{F}^*(t(\varphi))\}_{x/\lambda f}$$

$$U_{f_{L3}} = \mathcal{F}\{\beta' A e^{-j2\pi x\varphi_0/\lambda f}\, \mathcal{F}\{t(\varphi)_{x/\lambda f}\}\big|_{x/\lambda f}$$

$$= \beta' A_0 \int_{-\infty}^{\infty} e^{-j2\pi\left(\frac{x\varphi_0}{\lambda f}\right)} \left\{\int_{-\infty}^{\infty} t(\varphi)\, e^{-j2\pi\left(\frac{x\varphi}{\lambda f}\right)} d\varphi\right\} e^{-j2\pi x\varphi_L/\lambda f_L}\, dx$$

$$\cancel{= \beta' A_0 \int_{-\infty}^{\infty} e^{-j2\pi\frac{x\varphi_0}{\lambda f}}}$$

$$= \beta' A_0 \lambda f\, t\left(-\varphi_0 - \varphi_L \frac{f}{f_L}\right);$$ INVERTED MAGNEFIED IMAGE

SIMILARLY

$$U_{SL4} = B'A_0 \lambda f \; t^* \left(-\varphi_0 + \varphi_L \; f/f_L \right)$$

(8-9) TAKE σ_F X FORM OF $\mathscr{I}(x)$, DO

BOB MARKS

$\frac{6}{10}$

2-5) a) $p(x,y) = g(x,y) * \left[\text{comb}\left(\frac{x}{X}\right)\text{comb}\left(\frac{y}{Y}\right)\right]$

$\Rightarrow \mathcal{F}\{p(x,y)\} = \mathcal{F}\{g(x,y)\}\,\mathcal{F}\left\{\text{comb}\left(\frac{x}{X}\right)\text{comb}\left(\frac{y}{Y}\right)\right\}$

$P(f_x, f_y) = G(f_x, f_y)\,\mathcal{F}\left\{\text{comb}\left(\frac{y}{Y}\right)\right\}\mathcal{F}\left\{\text{comb}\left(\frac{y}{Y}\right)\right\}$

NOW: $\text{comb}\left(\frac{x}{X}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{X} - n\right)$

$= X \sum_{n=-\infty}^{\infty} \delta(x - Xn)$

FROM COOPER-McGILLEM; pg 129:

$X\,\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(x - nX)\right\} = X\,\frac{2\pi}{X}\sum_{n=-\infty}^{\infty} \delta\left(\omega_x - \frac{2\pi n}{X}\right) \quad \ni \omega_x = 2\pi f_x$ ✓ $k_x = 2\pi f_x$ is standard

$= \sum_{n=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}\right)$

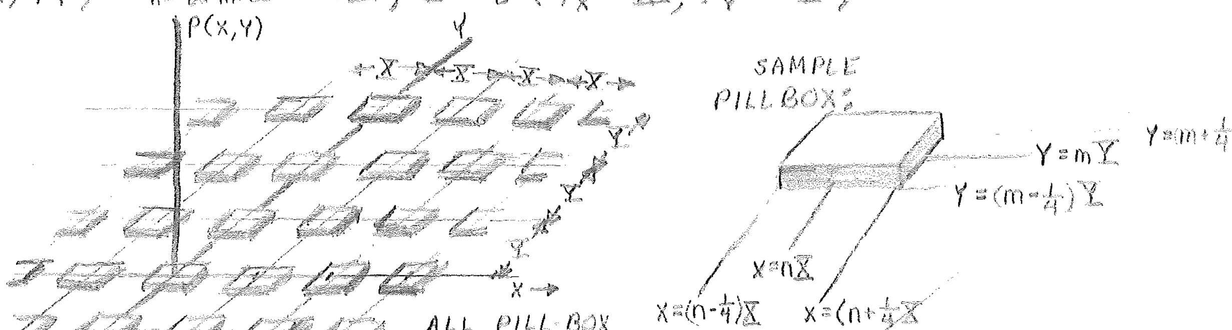
$\Rightarrow P(f_x, f_y) = G(f_x, f_y)\sum_{n=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}\right)\sum_{m=-\infty}^{\infty} \delta\left(f_y - \frac{m}{Y}\right)$

$= G(f_x, f_y)\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$

$P(f_x, f_y)$ ASSUMES ZERO VALUES FOR $f_x \neq \frac{n}{X}$ AND $f_y \neq \frac{m}{Y}$

$\therefore P(f_x, f_y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G\left(\frac{n}{X}, \frac{m}{Y}\right)\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$

b)



$\text{rect}\left(\frac{2x}{X}\right) = 1 \qquad -\frac{X}{4} \leq x \leq \frac{X}{4}$

$g(x,y) = \text{rect}\left(\frac{2x}{X}\right)\text{rect}\left(\frac{2y}{Y}\right)$; FROM 2-2 AND SIMILARITY THEM:

$\mathcal{F}\{g(x,y)\} = G(f_x, f_y) = \frac{XY}{4}\,\text{sinc}\left(\frac{f_x X}{2}\right)\text{sinc}\left(\frac{f_y Y}{2}\right)$

HENCE:

$P(f_x, f_y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{XY}{4}\,\text{sinc}\left(\frac{n}{2}\right)\text{sinc}\left(\frac{m}{2}\right)\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$

$P(f_x, f_y) = 0$ FOR ALL NON-ZERO EVEN m AND n

LET $n = 2p+1$; $m = 2q+1$

$\Rightarrow P(f_x, f_y) = \frac{XY}{4}\left[\delta(f_x, f_y) + \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \frac{4}{(2p+1)(2q+1)\pi^2}\sin\frac{(2p+1)\pi}{2}\cdot\sin\frac{(2q+1)\pi}{2}\,\delta\left(f_x - \frac{2p+1}{X}, f_y - \frac{2q+1}{Y}\right)\right]$

$= \frac{XY}{4}\left[\delta(f_x, f_y) + \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \frac{4}{(2p+1)(2q+1)\pi^2}(-1)^{p+q}\,\delta\left(f_x - \frac{2p+1}{X}, f_y - \frac{2q+1}{Y}\right)\right]$

( ) $p(x,y) = XY \sum_{n,m} \mathrm{rect}\left(\frac{2}{X}(x-nX)\right) \mathrm{rect}\left(\frac{2}{Y}(y-mY)\right)$.

$\therefore$ $p(x,y)$ is just the original function $\mathrm{rect}\left(\frac{2x}{X}\right)\mathrm{rect}\left(\frac{2y}{Y}\right)$ scaled by XY and translated to the lattice points $x = nX$, $y = mY$, $n, m = -\infty, \cdots, \infty$.

Note that $P(f_x, f_y)$ is just a sampled version of $G(f_x, f_y)$ in the frequency domain, while $p(x,y)$ is the translated version of $g(x,y)$ in the spatial domain. Compare these results with the sampling theorem.

( ) (2-9). In the development of the sampling theorem, take $H(f_x, f_y)$ (Eqn. 2-49) to be

$$H(f_x, f_y) = \mathrm{circ}\left(\frac{\rho}{B}\right).$$

Then the impulse response of this filter, which is the same as its inverse Fourier transform (Fourier-Bessel transform) is (see p. 13).

$$\mathcal{B}^{-1}\left\{\mathrm{circ}\left(\frac{\rho}{B}\right)\right\} = \mathcal{B}\left\{\mathrm{circ}\left(\frac{\rho}{B}\right)\right\} = B^2 \frac{J_1(2\pi Br)}{Br}.$$

$$\mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right) g(x,y) = XY \sum_{n,m} g(nX, mY)\, \delta(x-nX, y-mY)$$

$$g(x,y) = \left[XY \sum_{n,m} g(nX, mY)\, \delta(x-nX, y-mY)\right] ** B^2 \frac{J_1(2\pi Br)}{Br}$$

or $g(x,y) = XY\,B^2 \sum_{n,m} g(nX, mY) \left\{ \frac{J_1\left[2\pi B\sqrt{(x-nX)^2+(y-mY)^2}\right]}{B\sqrt{(x-nX)^2+(y-mY)^2}} \right\}$

Now the maximum spacing for X and Y, each, is $\frac{1}{2B}$. Then, letting $X = 1/2B$, $Y = 1/2B$, we get

$$g(x,y) = \sum_{n,m} g\left(\frac{n}{2B}, \frac{m}{2B}\right) \left\{ \frac{J_1\left[2\pi B\sqrt{\left(x-\frac{n}{2B}\right)^2+\left(y-\frac{m}{2B}\right)^2}\right]}{4B\sqrt{\left(x-\frac{n}{2B}\right)^2+\left(y-\frac{m}{2B}\right)^2}} \right\}.$$

(2.12). Sampling in the x and y directions must occur at the respective rates $1/2B_x$, $1/2B_y$ per meter. Hence, in an area $2X \times 2Y = 4XY$ m², we must have $\frac{4XY}{\frac{1}{4B_xB_y}} = 16 B_x B_y X Y$ for each part (real and imaginary), making $32 B_x B_y X Y$ in all.

We know in general, that we need all discrete data points to recover exactly the given function. If the sampled values $g\left(\frac{n}{2B_x}, \frac{m}{2B_y}\right)$ are sufficiently small outside of the given area $|x| \le X$, $|y| \le Y$, then the approximation will be good.

It should be kept in mind, however, that all space-limited functions have infinite bandwidth, so that only approx[illegible] that a function is both space and b[illegible]

$$U_o(x,y) \longrightarrow \boxed{\text{Low-pass filter}} \longrightarrow U_i \longrightarrow \boxed{\text{Sampling}} \longrightarrow U_o'(x,y)$$

U_i is band-limited after U_o is passed through the low-pass filter (the imaging system). The impulse response of the low-pass filter is

$$\mathcal{F}^{-1}\left\{ \operatorname{rect}\left(\frac{f_x}{2B_x}\right) \operatorname{rect}\left(\frac{f_y}{2B_y}\right) \right\} = 4B_x B_y \operatorname{sinc}(2B_x x)\operatorname{sinc}(2B_y y)$$

$$\therefore \quad U_i(x,y) = 4B_xB_y \iint_{-\infty}^{\infty} U_o(\xi,\eta)\, \operatorname{sinc}(2B_x(x-\xi))\operatorname{sinc}(2B_y(y-\eta))\, d\xi\, d\eta$$

The sampling operation simply multiplies $U_i(x,y)$ by the function $\operatorname{comb}\left(\frac{x}{X}\right)\operatorname{comb}\left(\frac{y}{Y}\right)$, where we take $X = \frac{1}{2B_x}$, $Y = \frac{1}{2B_y}$

$$\therefore \quad \operatorname{comb}(2B_x x)\operatorname{comb}(2B_y y) = \frac{1}{4B_xB_y}\sum_{m,n} \delta\left(x - \frac{n}{2B_x},\, y - \frac{m}{2B_y}\right)$$

and

$$U_o'(x,y) = \sum_{m,n} \delta\left(x - \frac{n}{2B_x},\, y - \frac{m}{2B_y}\right) \cdot \iint_{-\infty}^{\infty} U_o(\xi,\eta)\operatorname{sinc}(2B_x(x-\xi))\operatorname{sinc}(2B_y(y-\eta))\, d\xi\, d\eta$$

$$= \sum_{m,n} \delta\left(x - \frac{n}{2B_x},\, y - \frac{m}{2B_y}\right) \cdot \iint_{-\infty}^{\infty} U_o(\xi,\eta)\operatorname{sinc}(n - 2B_x\xi)\operatorname{sinc}([illegible])\, d\xi\, d\eta$$

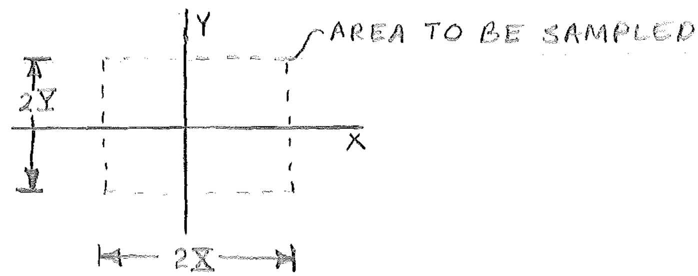
(Note that due to the delta function we may just replace $x = \frac{n}{2B_x}$, and [illegible] $y = \frac{m}{2B_y}$ [illegible].)

BOB MARKS

$\frac{20}{20}$

2-12) LET $g(x,y)$ BE THAT "CERTAIN COMPLEX FUNCTION"

$$G(f_x, f_y) = 0, \text{ FOR } |f_x| \leq B_x, |f_y| \leq B_y$$



FUNCTION IS BAND LIMITED. SAMPLING THEM. DICTATES SAMPLING MUST TAKE PLACE AT INTERVALS OF $\frac{1}{2B_x}$ IN X DIRECTION AND $\frac{1}{2B_y}$ IN Y DIRECTION.

NUMBER OF INTERVALS IN X DIRERECTION (S_x) IN AREA $|x| \leq X$:

$$S_x = \frac{2X}{1/2B_x} = 4XB_x$$

AND

$$S_y = \frac{2Y}{1/2B_y} = 4YB_y$$

TOTAL NUMBER ON SAMPLES:

$$S = S_x S_y = 16XYB_xB_y$$

EACH SAMPLE IS COMPLEX, AND THUS REQUIRES NO LESS THAN TWO REAL NUMBERS PER SAMPLE

⇒ FUNCTION CAN BE SPECIFIED BY $32XYB_xB_y$ REAL NUMBERS.

ALTHOUGH $g(x,y)$ HAS A FINITE SPECTRA, SAMPLING ONLY OVER $x \leq X$ AND $y \leq Y$ IMPLIES $g(x,y)$ IS ZERO OUTSIDE X. AND Y WHICH FURTHER IMPLIES A NON-FINITE SPECTRUM. ✓ ie LIMITING $g(x,y)$ TO THE ABOVE INTERVALS CHANGES ITS SPECTRA FROM FINITE TO INFINITE (HEISENBURG'S UNCERTAINTY PRINCIPAL) THIS SAMPLING WOULD YIELD A GOOD RECONSTRUCTION IF $g(x,y)$ IS VERY SMALL OUTSIDE THE SPACIAL RECTANGLE AS OPPOSED TO $g(x,y)$ INSIDE.

BOB MARKS

2-13) $U_i(x,y) = S[U_o(x,y)]$

SYSTEM: LOW PASS FILTER

$$H(f_x,f_y) = \mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)$$

$$\Rightarrow\ h(x,y) = 4B_xB_y\,\mathrm{sinc}(2B_x x)\,\mathrm{sinc}(2B_y y)$$

NOW: $U_i(x,y) = U_o(x,y) \circledast h(x,y)$

$$= \int_{-\infty}^{\infty}\!\!\int U_o(\varepsilon,\eta)\,h[(x-\varepsilon),(y-\eta)]\,d\varepsilon\,d\eta$$

$$= 4B_xB_y\int_{-\infty}^{\infty}\!\!\int U_o(\varepsilon,\eta)\,\mathrm{sinc}[2B_x(x-\varepsilon)]\,\mathrm{sinc}[2B_y(y-\eta)]\,d\varepsilon\,d\eta$$

$U_o'(x,y)$ = SAMPLED $U_i(x,y)$

$$\Rightarrow U_o'(x,y) = \mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)U_i(x,y)$$

FOR MINIMUM NUMBER OF SAMPLING POINTS:

$$X = \frac{1}{2B_x}\ ;\quad Y = \frac{1}{2B_y}$$

$$\Rightarrow U_o'(x,y) = \mathrm{comb}(2B_x x)\,\mathrm{comb}(2B_y y)\,U_i(x,y)$$

$$= \sum_{n,m=-\infty}^{\infty}\delta(2B_x x - n, 2B_y y - m)\,U_i(x,y)$$

$$= \frac{1}{4B_xB_y}\sum_{m=n=-\infty}^{\infty}\delta\left(x-\frac{n}{2B_x}, y-\frac{m}{2B_y}\right)U_i(x,y)$$

$$= \frac{1}{4B_xB_y}\sum_{n=m=-\infty}^{\infty}\delta\left(x-\frac{n}{2B_x}, y-\frac{m}{2B_y}\right)$$

$$\cdot 4B_xB_y\int_{-\infty}^{\infty}\!\!\int U_o(\varepsilon,\eta)\,\mathrm{sinc}[2B_x(x-\varepsilon)]\,\mathrm{sinc}[2B_y(y-\eta)]\,d\varepsilon\,d\eta$$

$$= \sum_{m=n=-\infty}^{\infty}\delta\left(x-\frac{n}{2B_x}, y-\frac{m}{2B_y}\right)\int_{-\infty}^{\infty}\!\!\int U_o(\varepsilon,\eta)\,\mathrm{sinc}[2B_x(x-\varepsilon)]\,\mathrm{sinc}[2B_y(y-\eta)]\,d\varepsilon\,d\eta$$

$U_o'(x,y)$ ASSUMES NON-ZERO VALUES ONLY AT $x = \frac{n}{2B_x}$ AND $y = \frac{m}{2B_y}$

$$\Rightarrow U_o'(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\!\!\int U_o(\varepsilon,\eta)\,\mathrm{sinc}(n-2B_x\varepsilon)\,\mathrm{sinc}(m-2B_y\eta)\,d\varepsilon\,d\eta\right]$$
$$\delta\left(x-\frac{n}{2B_x}, y-\frac{m}{2B_y}\right)$$

HENCE, PASSING U_o THRU THE LOW PASS FILTER YIELDS U_i WHICH IS LIMITED IN BANDWIDTH. SAMPLING U_i AT MINIMUM RATE YIELDS U_o'. PASSING U_o' AGAIN THRU THE FILTER WILL AGAIN PRODUCE U_i, IN THAT CRITERIA FOR THE SAMPLING THEM. IS MET

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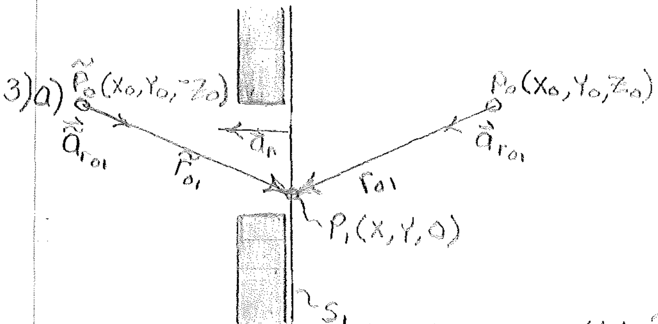
3-3) a)



$$G_+(P_1) = \frac{\exp(jkr_{01})}{r_{01}} + \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}$$

NORMAL DIRECTIONAL DERIVATIVE: $\frac{\delta G_+(P_1)}{\delta n}$

$$\frac{\delta G_+(P_1)}{\delta n} = \vec{a}_n \cdot \nabla G_+(P_1)$$
$$= \vec{a}_n \cdot \nabla \left[\frac{\exp(jkr_{01})}{r_{01}} + \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}} \right]$$

NOW: $\nabla\left(\frac{1}{r_{01}} e^{jkr_{01}}\right) = \vec{a}_{r_{01}} \frac{\delta}{\delta r_{01}} \left[\frac{1}{r_{01}} e^{jkr_{01}}\right]$
$$= \vec{a}_{r_{01}} \left[jk - \frac{1}{r_{01}}\right]\left[\frac{1}{r_{01}} e^{jkr_{01}}\right]$$

SIMILARLY: $\nabla\left(\frac{1}{\tilde{r}_{01}} e^{jk\tilde{r}_{01}}\right) = \tilde{\vec{a}}_{r_{01}} \left[jk - \frac{1}{\tilde{r}_{01}}\right]\left[\frac{1}{\tilde{r}_{01}} e^{jk\tilde{r}_{01}}\right]$

$$\Rightarrow \frac{\delta G_+(P_1)}{\delta n} = \vec{a}_n \cdot \vec{a}_{r_{01}} \left(jk - \frac{1}{r_{01}}\right)\left(\frac{1}{r_{01}} e^{jkr_{01}}\right) + \vec{a}_n \cdot \tilde{\vec{a}}_{r_{01}} \left(jk - \frac{1}{\tilde{r}_{01}}\right)\left(\frac{1}{\tilde{r}_{01}} e^{jk\tilde{r}_{01}}\right)$$

@ P_1, $\vec{a}_n \cdot \vec{a}_{r_{01}} = -\vec{a}_n \cdot \tilde{\vec{a}}_{r_{01}}$

AND $|r_{01}| = |\tilde{r}_{01}|$

$$\therefore \frac{\delta G_+(P_1)}{\delta n} = \vec{a}_n \cdot \vec{a}_{r_{01}} \left(jk - \frac{1}{r_{01}}\right)\left(\frac{1}{r_{01}} e^{jkr_{01}}\right) - \vec{a}_n \cdot \vec{a}_{r_{01}} \left(jk - \frac{1}{r_{01}}\right)\left(\frac{1}{r_{01}} e^{jkr_{01}}\right) = 0$$

b) $U(P_0) = \frac{1}{4\pi} \iint_{S_1} \left(G \frac{\delta U}{\delta n} - U \frac{\delta G}{\delta n} \right) dS$

FROM (a) $\delta G / \delta n = 0$ @ P_1 ON S_1

$$\Rightarrow U(P_0) = \frac{1}{4\pi} \iint_{S_1} G \frac{\delta U}{\delta n} dS$$
$$= \frac{1}{4\pi} \iint_{S_1} \left(\frac{2}{r_{01}} e^{jkr_{01}}\right) \frac{\delta U}{\delta n} dS$$

BECAUSE $r_{01} = \tilde{r}_{01}$ @ $P_1 \Rightarrow G_+(P_1) = \frac{2}{r_{01}} e^{jkr_{01}}$

$$\therefore U(P_0) = \frac{1}{2\pi} \iint_{S_1} \left(\frac{1}{r_{01}} e^{jkr_{01}}\right) \frac{\delta U(P_1)}{\delta n} dS$$

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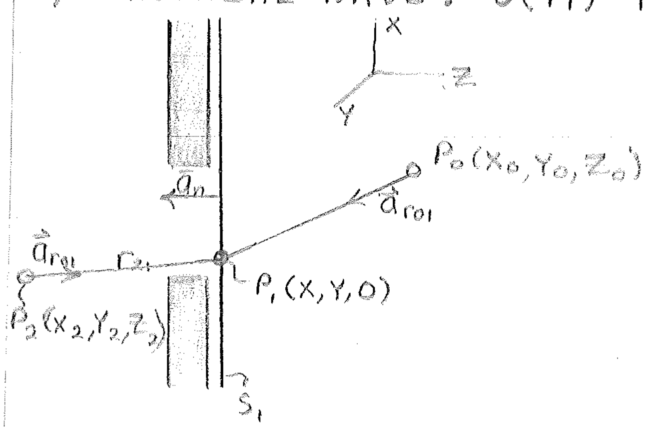
c) SPHERICAL WAVE: $U(P_1) = \frac{A}{r_{21}} \exp(jkr_{12})$



AS IN (b); $\frac{\delta U(P_1)}{\delta n} = \vec{a}_n \cdot \nabla U(P_1)$

$$= \vec{a}_n \cdot \vec{a}_{r_{21}} \left(jk - \frac{1}{r_{21}}\right) \frac{A}{r_{21}} e^{jkr_{21}}$$

LET $\vec{a}_x$, $\vec{a}_y$, AND $\vec{a}_z$ BE UNIT VECTORS IN X, Y, AND Z DIRECTION RESPECTIVELY;

$$\vec{a}_{r_{21}} = \left[(X-X_2)^2 + (Y-Y_2)^2 + Z_2^2\right]^{-\frac{1}{2}} \left[(X_2 - X)\vec{a}_x + (Y_2 - Y)\vec{a}_y + Z_2 a_z\right]$$
$$= \frac{1}{r_{21}} \left[(X_2 - X)\vec{a}_x + (Y_2 - Y)\vec{a}_y + Z_2 \vec{a}_z\right]$$

AND: $\vec{a}_n = -\vec{a}_z$

$$\Rightarrow \vec{a}_{r_{21}} \cdot \vec{a}_n = -\frac{1}{r_{21}} Z_2$$
$$\Rightarrow \frac{\delta U(P_1)}{\delta n} = -\frac{AZ_2}{r_{21}^2}\left(jk - \frac{1}{r_{21}}\right) e^{jkr_{21}}$$

FROM (b);

$$U(P_0) = \frac{1}{2\pi} \int_{S_1}\!\!\int \left(\frac{1}{r_{01}} e^{jkr_{01}}\right) \frac{\delta U(P_1)}{\delta n} \, dS$$
$$= \frac{-1}{2\pi} \int_{S_1}\!\!\int \left(\frac{1}{r_{01}} e^{jkr_{01}}\right) \frac{AZ_2}{r_{21}^2}\left(jk - \frac{1}{r_{21}}\right) e^{jkr_{21}} \, dS$$
$$= \frac{AZ_2}{2\pi} \int_{S_1}\!\!\int \frac{1}{r_{01} r_{21}^2}\left(jk - \frac{1}{r_{21}}\right) e^{jk(r_{21}+r_{01})} \, dS$$

FOR $|jk| >> |1/r_{21}|$

$$U(P_0) = \frac{jkAZ_2}{2\pi} \int_{S_1}\!\!\int \frac{1}{r_{01} r_{21}^2} e^{jk(r_{21}+r_{01})} \, dS$$

$\ni r_{01} = \left[(X_0 - X)^2 + (Y_0 - Y)^2 + Z_0^2\right]^{\frac{1}{2}}$;

$r_{21} = \left[(X_2 - X)^2 + (Y_2 - Y)^2 + Z_2^2\right]^{\frac{1}{2}}$

$dS = dX\,dY$; INTEGRATION OVER "WINDOW"

Fourier Optics : Solution of 3-3.

(3-3) (a) $G_+(P_1) = \dfrac{\exp(jkr_{01})}{r_{01}} + \dfrac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}$

on S_1, $G_+(P_1) = \dfrac{2\exp(jkr_{01})}{r_{01}}$

$$\frac{\partial G_+}{\partial n} = \bar{a}_n \cdot \nabla G_+(P_1) = \bar{a}_n \cdot \left(\bar{a}_{r_{01}}\left(jk - \frac{1}{r_{01}}\right) + \bar{a}_{\tilde{r}_{01}}\left(jk - \frac{1}{r_{01}}\right)\right) \cdot \frac{e^{jkr_{01}}}{r_{01}}$$

But $\bar{a}_n \cdot \bar{a}_{r_{01}} = -\bar{a}_n \cdot \bar{a}_{\tilde{r}_{01}}$ on S_1. Thus,

$\boxed{\dfrac{\partial G_+}{\partial n} = 0}$ on S_1.

(b). $$U(P_0) = \frac{1}{4\pi}\int_{S_1} \frac{2\exp(jkr_{01})}{r_{01}} \cdot \frac{\partial U(P_1)}{\partial n}\, ds.$$

Must now give the normal derivative $\dfrac{\partial U}{\partial n}$ on S_1

(c) $U(P_0) = \dfrac{1}{2\pi}\displaystyle\int_{\Sigma} \frac{\exp(jkr_{01})}{r_{01}} \cdot \frac{\partial U}{\partial n}(P_1)\, ds$, where $\dfrac{\partial U}{\partial n} = 0$ on S_1 except on Σ.

For spherical wave diverging from P_2 : $U(P_1) = \dfrac{A\exp(jkr_{21})}{r_{21}}$

$$\frac{\partial U(P_1)}{\partial n} = \bar{a}_n \cdot A\,\frac{\partial}{\partial r_{21}}\left(\frac{\exp(jkr_{21})}{r_{21}}\right)\bar{a}_{r_{21}} = \bar{a}_n \cdot \bar{a}_{r_{21}}\, A\left[jk - \frac{1}{r_{21}}\right]\frac{e^{jkr_{21}}}{r_{21}}$$

If $r_{21} >> \lambda$ or $\frac{1}{r_{21}} << \frac{2\pi}{\lambda} = k$, then $k >> \frac{1}{r_{21}}$

So that $\frac{\partial U}{\partial n}(P_1) \approx \bar{a}_n \cdot \bar{a}_{r_{21}}\, jkA \frac{e^{jkr_{21}}}{r_{21}}$ and

$$U(P_0) = \frac{jkA}{2\pi} \int_{\Sigma'} \bar{a}_n \cdot \bar{a}_{r_{21}} \frac{\exp jk(r_{01}+r_{21})}{r_{01}\, r_{21}}\, ds$$

$$= -\frac{A}{j\lambda} \int_{\Sigma} \bar{a}_n \cdot \bar{a}_{r_{21}} \frac{\exp jk(r_{01}+r_{21})}{r_{01}\, r_{21}}\, ds.$$

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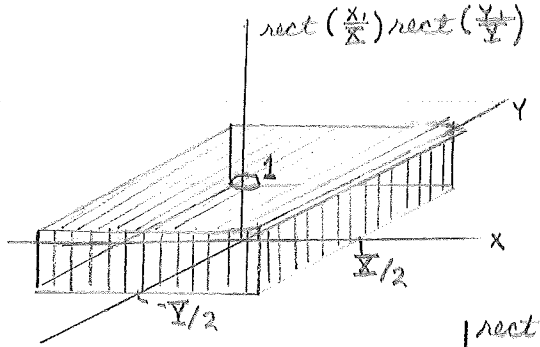
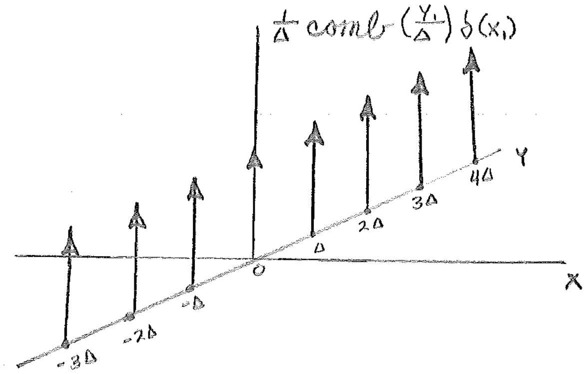
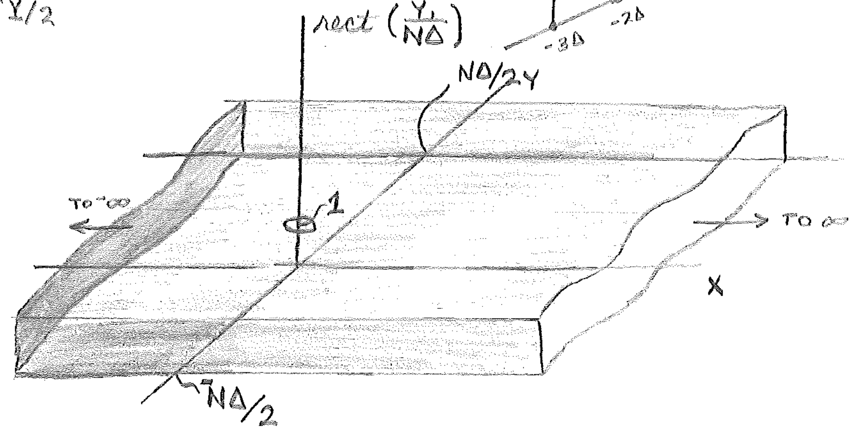
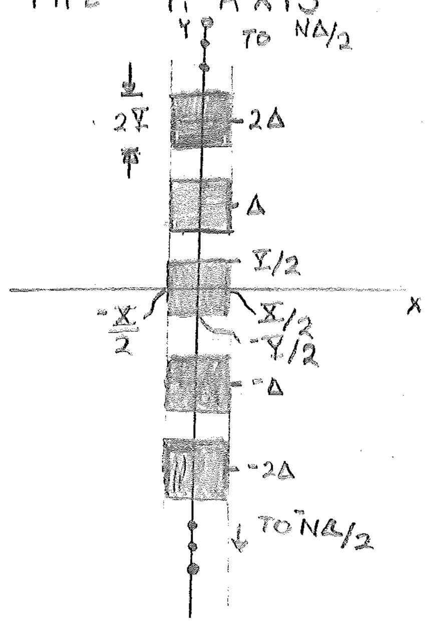
4·2) $t_1(x_1,y_1) = \left\{ \left[\mathrm{rect}\left(\frac{x_1}{X}\right) \mathrm{rect}\left(\frac{y_1}{Y}\right) \right] * \left[\frac{1}{\Delta} \mathrm{comb}\left(\frac{y_1}{\Delta}\right) \delta(x_1) \right] \right\} \mathrm{rect}\left(\frac{y_1}{N\Delta}\right)$

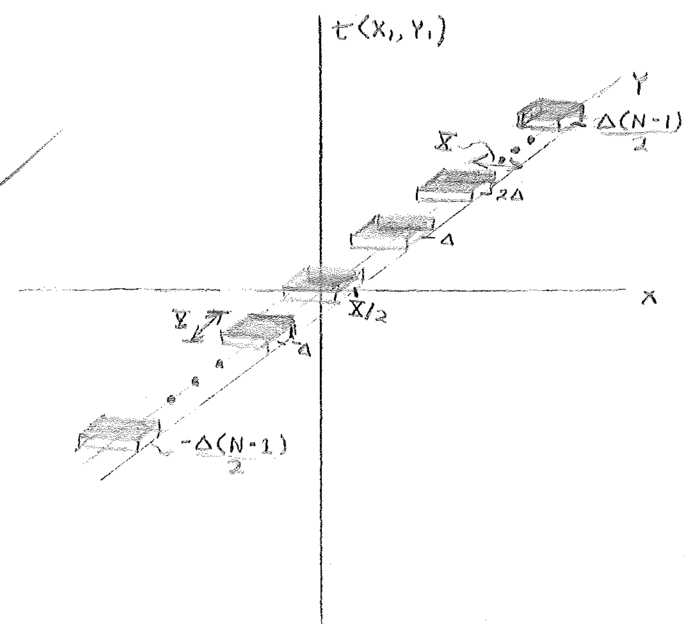






t(x₁, y₁) IS THUS REPRESENTED BY N UNIT AMPLITUDE "PILL BOXES" SPACED EVERY Δ UP THE Y₁ AXIS





b) $t(x_1,y_1) = rect\left(\frac{x_1}{X}\right) \sum_{n=-(N-1)/2}^{\frac{N-1}{2}} rect\left(\frac{y_1 - n\Delta}{Y}\right)$; THEN FROM SCALING AND "TIME" SHIFT THEOREMS

$$\Rightarrow T(f_x,f_y) = XY \, sinc(Xf_x)\, sinc(Yf_y) \sum_{n=-(N-1)/2}^{\frac{N-1}{2}} e^{-j2\pi f_y n\Delta}$$

FRAUNHOFFER APPROXIMATION;

$$U(x_0,y_0) = \frac{-j}{\lambda z} e^{jkz} exp\left[\frac{jk}{2z}(x_0^2+y_0^2)\right] \int_{-\infty}^{\infty}\int U(x_1,y_1)\, exp\left[\frac{-jk}{z}(x_0x_1+y_0y_1)\right] dx_1 dx_2$$

BUT $\int_{-\infty}^{\infty}\int U(x_1,y_1)\, exp\left[\frac{-jk}{z}(x_0x_1+y_0y_1)\right] dx_1 dy_1 = \mathcal{F}[U(x_1,y_1)]\Big|_{f_x = x_0/\lambda z}^{f_y = y_0/\lambda z}$

NOW $U_i = Ae^{jkz} \Rightarrow U(x_1,y_1) = A\,t(x_1,y_1)$

$$\therefore \mathcal{F}[U(x_1,y_1)] = XYA\, sinc(Xf_x)\, sinc(Yf_y) \sum_{n=-(N-1)/2}^{\frac{N-1}{2}} e^{-j2\pi f_y n\Delta}$$

BUT $\sum_{n=-(N-1)/2}^{(N-1)/2} e^{-j2\pi f_y n\Delta}$ IS A SUMMATION OF N TERMS OF A GEOMETRIC SERIES, WITH A COMMON RATIO $= \frac{a_{n+1}}{a_n} = r = e^{-j2\pi f_y\Delta}$

Good boy.

$$\Rightarrow \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} e^{-j2\pi f_y \Delta n} = a_1\left[\frac{1-r^N}{1-r}\right]$$

$$= e^{j2\pi f_y\left(\frac{N-1}{2}\right)\Delta}\left[\frac{1-e^{-j2\pi f_y N\Delta}}{1-e^{-j2\pi f_y\Delta}}\right]$$

$$= e^{j2\pi f_y\left(\frac{N-1}{2}\right)\Delta}\left[\frac{e^{-j\pi f_y N\Delta}}{e^{-j\pi f_y\Delta}}\right]\left[\frac{e^{j\pi f_y N\Delta} - e^{-j\pi f_y N\Delta}}{e^{j\pi f_y\Delta} - e^{-j\pi f_y\Delta}}\right]$$

$$= \frac{sin(\pi f_y N\Delta)}{sin(\pi f_y\Delta)}$$

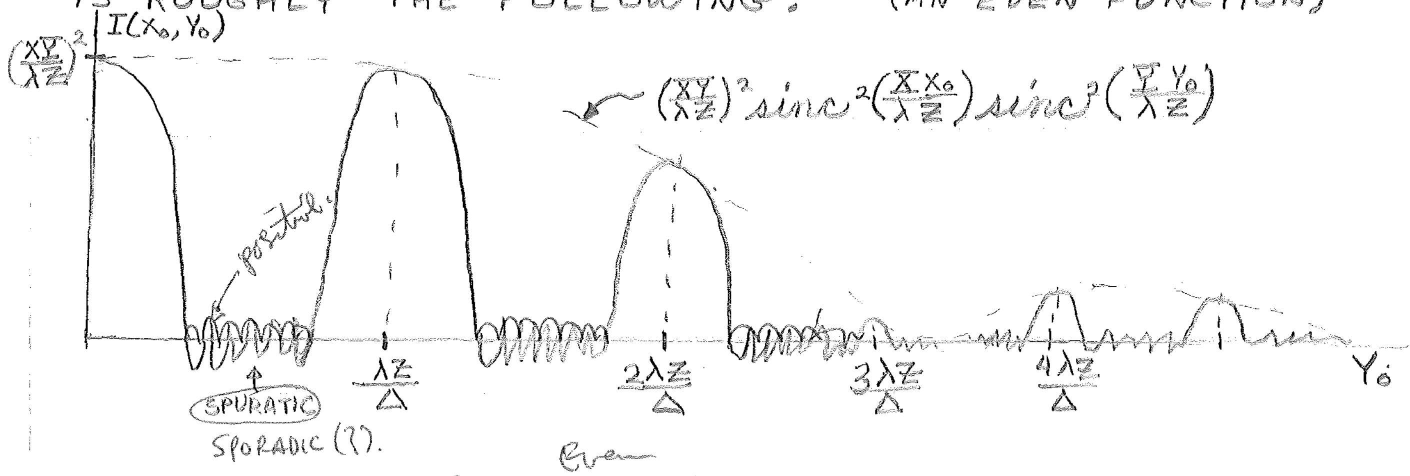
$$\Rightarrow \mathcal{F}[U(x_1,y_1)] = XYA\, sinc(Xf_x)\, sinc(Yf_y)\frac{sin(\pi f_y N\Delta)}{sin(\pi f_y\Delta)}$$

NOW $f_x = \frac{x_0}{\lambda z}$ AND $f_y = \frac{y_0}{\lambda z}$

$$\therefore U(x_0,y_0) = \frac{-j}{\lambda z} e^{jkz} exp\left\{\frac{jk}{2z}(x_0^2+y_0^2)\right\} XYA\, sinc\left(\frac{Xx_0}{\lambda z}\right) sinc\left(\frac{Yy_0}{\lambda z}\right) sin\left(\frac{\pi y_0 N\Delta}{\lambda z}\right) \Big/ sin\left(\frac{\pi y_0\Delta}{\lambda z}\right)$$

$$\Rightarrow I(x_0,y_0) = |U(x_0,y_0)|^2 = \left(\frac{XY}{\lambda z}\right)^2 sinc^2\left(\frac{Xx_0}{\lambda z}\right) sinc^2\left(\frac{Yy_0}{\lambda z}\right)\frac{sin^2\left(\frac{\pi y_0 N\Delta}{\lambda z}\right)}{sin^2\left(\frac{\pi y_0\Delta}{\lambda z}\right)}$$

c) AT A GIVEN Z AND X_0 THE INTENSITY (GRAPHICALLY) IS ROUGHLY THE FOLLOWING: (AN EVEN FUNCTION)



THE REDUCTION OF ~~HIGHER~~ Even ORDER PULSES MAY BE ACCOMPLISHED IN TWO MANNERS

1) DECREASE THE VALUE OF Δ, THUS MAKING THE INTERVALS BETWEEN ADJACENT PULSES LARGER.
2) INCREASE THE VALUE OF $\mathcal{Y}$, MAKING A FASTER FALL OFF TIME IN THE DAMPING TERM $sinc^2\left(\frac{\mathcal{Y} Y_0}{\lambda Z}\right)$

BECAUSE $\mathcal{Y} < \Delta$, THE LIMITING CASE WOULD BY $\mathcal{Y} = \Delta$, IN WHICH CASE THE APERTURE WOULD BE A LONG NARROW STRIP

4-6) $t(x_1, y_1) = \frac{1}{2}(1 + m \cos 2\pi f_0 x_1)$; $U_i = A e^{jkz}$

$$\Rightarrow T(f_x, f_y) = \frac{1}{2}\left[\delta(f_x, f_y) + \frac{m}{2}\{\delta(f_x + f_0, f_y) + \delta(f_x - f_0, f_y)\}\right]$$

a) FRESNEL DIFFRACTION

$$U(x_0, y_0) = \frac{1}{j\lambda z}\exp(jkz)\exp\left[\frac{jk}{2z}(x_0^2 + y_0^2)\right]$$
$$\int_{-\infty}^{\infty}\!\!\int \left\{U(x_1, y_1)\exp\left[\frac{jk}{2z}(x_1^2 + y_1^2)\right]\exp\left[\frac{-jk}{z}(x_0 x_1 + y_0 y_1)\right] dx_1\, dy_1\right.$$

BUT $\int_{-\infty}^{\infty}\!\!\int U(x_1, y_1)\exp\left[\frac{jk}{2z}(x_1^2 + y_1^2)\right]\exp\left[\frac{-jk}{z}(x_0 x_1 + y_0 y_1)\right] dx_1\, dy_1$

$$= \mathcal{F}\left[U(x_1, y_1)\exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\}\right]_{f_x = x_0/\lambda z}^{f_y = y_0/\lambda z}$$
$$= \mathcal{F}[U(x_1, y_1)] \circledast \mathcal{F}\left[\exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\}\right]_{f_x = kx_0/2\pi z}^{f_y = ky_0/2\pi z}$$

$$\mathcal{F}[U(x_1, y_1)] = \frac{A}{2}\left[\delta(f_x, f_y) + \frac{m}{2}\{\delta(f_x + f_0, f_y) + \delta(f_x - f_0, f_y)\}\right]$$

$$\mathcal{F}\left[\exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\}\right] = \mathcal{F}\left[\exp\left(\frac{jkx_1^2}{2z}\right)\right]\mathcal{F}\left[\exp\left(\frac{jky_1^2}{2z}\right)\right]$$

$$\mathcal{F}\left[\exp\left(\frac{jkx_1^2}{2z}\right)\right] = \int_{-\infty}^{\infty}\exp\left[\frac{jk}{2z}x_1^2\right]\exp[-j2\pi f_x x_1]\, dx_1$$
$$= \int_{-\infty}^{\infty}\exp j\left(\frac{k}{2z}x_1^2 - 2\pi f_x x_1\right) dx_1$$
$$= \int_{-\infty}^{\infty}\exp j\left[\left(\sqrt{\frac{k}{2z}}x_1 - \pi f_x\sqrt{\frac{2z}{k}}\right)^2 - \frac{2\pi^2 f_x^2 z}{k}\right] dx_1$$
$$= \exp\left(\frac{-j2\pi^2 f_x^2 z}{k}\right)\int_{-\infty}^{\infty}\exp\left(\sqrt{\frac{k}{2z}}x_1 - \pi f_x\sqrt{\frac{2z}{k}}\right)^2 dx_1$$

LET $U = \sqrt{\frac{k}{2z}}x_1 - \pi f_x\sqrt{\frac{2z}{k}} \Rightarrow dx_1 = \sqrt{\frac{2z}{k}}\, dU$

$$\Rightarrow \mathcal{F}\left[\exp\left(\frac{jkx_1^2}{2z}\right)\right] = \sqrt{\frac{2z}{k}}\exp\left(\frac{-j2\pi^2 f_x^2 z}{k}\right)\int_{-\infty}^{\infty}\exp(U^2)\, dU$$
$$= \sqrt{\frac{2z}{k}}\exp\left(\frac{-j2\pi^2 f_x^2 z}{k}\right)\sqrt{\frac{\pi}{2}}(j + 1)$$
$$= \sqrt{\frac{2\pi z}{k}}\exp\left(\frac{-j2\pi^2 f_x^2 z}{k}\right)\exp\left(\frac{j\pi}{4}\right)$$

$$\Rightarrow \mathcal{F}\left[\exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\}\right] = \frac{2\pi z}{k}\exp\left[-j\left\{\frac{2\pi^2 z}{k}(f_x^2 + f_y^2) - \frac{\pi}{2}\right\}\right]$$

$$\mathcal{F}\left[U(x, y_1)\exp\left\{\frac{jk}{2z}(x_1^2 + y_1^2)\right\}\right] = \frac{A}{2}\left[\delta(f_x, f_y) + \frac{m}{2}\{\delta(f_x + f_0, f_y) + \delta(f_x - f_0, f_y)\}\right]$$
$$\circledast\left[\frac{2\pi z}{k}\exp\left\{j\left(\frac{\pi}{2} - \frac{2\pi^2 z}{k}(f_x^2 + f_y^2)\right)\right\}\right]$$
$$= \frac{\pi A z}{k}\int_{-\infty}^{\infty}\!\!\int\left[\delta(\varepsilon, \eta) + \frac{m}{2}\{\delta(\varepsilon + f_0, \eta) + \delta(\varepsilon - f_0, \eta)\}\right]$$
$$\exp\left[j\left\{\frac{\pi}{2} - \frac{2\pi^2 z}{k}\{(f_x - \varepsilon)^2 + (f_y - \eta)^2\}\right\}\right] d\varepsilon\, d\eta$$
$$= \frac{\pi A z}{k}\left[\exp j\left\{\frac{\pi}{2} - \frac{2\pi^2 z}{k}(f_x^2 + f_y^2)\right\} + \frac{m}{2}\left\{\exp j\left\{\frac{\pi}{2} - \frac{2\pi^2 z}{k}[(f_x + f_0)^2 + f_y^2]\right\}\right.\right.$$
$$\left.\left. + \exp j\left\{\frac{\pi}{2} - \frac{2\pi^2 z}{k}[(f_x - f_0)^2 + f_y^2]\right\}\right\}\right]$$
$$= \frac{\pi A z}{k}\exp j\left\{\frac{\pi}{2} - \frac{2\pi^2 z}{k}f_y^2\right\}\left[\exp\left(-j\frac{2\pi^2 z}{k}f_x^2\right) + \frac{m}{2}\left(\exp\left\{\frac{-j2\pi^2 z}{k}(f_x + f_0)^2\right\}\right.\right.$$
$$\left.\left. + \exp\left\{\frac{-j2\pi^2 z}{k}(f_x - f_0)^2\right\}\right)\right]$$

$$\mathcal{F}\left[U(x_1,y_1)\exp\left\{\frac{jk}{2z}(x_1^2+y_1^2)\right\}\right]_{\substack{f_Y = kY_0/2\pi z \\ f_X = kX_0/2\pi z}}$$

$$= \frac{\pi A z}{k}\exp j\left\{\frac{\pi}{2} - \frac{2\pi^2 z}{k}\left(\frac{kY_0}{2\pi z}\right)^2\right\}\left[\exp\left\{\frac{-j2\pi^2 z}{k}\left(\frac{kX_0}{2\pi z}\right)^2\right\} + \frac{m}{2}\left\{\exp\left(\frac{-j2\pi^2 z}{k}\left(\frac{kX_0}{2\pi z}+f_0\right)^2\right\} + \exp\left\{\frac{-j2\pi^2 z}{k}\left(\frac{kX_0}{2\pi z}-f_0\right)^2\right\}\right\}\right]$$

$$= \frac{\pi A z}{k}\exp j\left\{\frac{\pi}{2} - \frac{kY_0^2}{2z}\right\}\exp\left\{\frac{-jkX_0^2}{2z}\right\}\left[1+\frac{m}{2}\left\{\exp\left\{\frac{-j2\pi^2 z}{k}\left(\frac{2kX_0 f_0}{2\pi z}+f_0^2\right)\right\} + \exp\left\{\frac{-j2\pi^2 z}{k}\left(\frac{-2kX_0 f_0}{2\pi z}+f_0^2\right)\right\}\right\}\right]$$

$$= \frac{\pi A z}{k}\exp j\left\{\frac{\pi}{2} - \frac{k}{2z}(X_0^2+Y_0^2)\right\}\left[1+\frac{m}{2}\exp\left\{\frac{-j2\pi^2 z f_0^2}{k}\right\}\left\{\exp(-j2\pi X_0 f_0) + \exp(j2\pi X_0 f_0)\right\}\right]$$

$$= \frac{\pi A z}{k}\exp j\left\{\frac{\pi}{2} - \frac{k}{2z}(X_0^2+Y_0^2)\right\}\left[1+m\exp\left(\frac{-j2\pi^2 z f_0^2}{k}\right)\cos 2\pi X_0 f_0\right]$$

$$\therefore U(X_0,Y_0) = \frac{k}{j2\pi z}\exp(jkz)\exp\left\{\frac{jk}{z}(X_0^2+Y_0^2)\right\}\frac{\pi A z}{k}\exp j\left\{\frac{\pi}{2} - \frac{k}{2z}(X_0^2+Y_0^2)\right\}\left[1+m\exp\left(\frac{-j2\pi^2 z f_0^2}{k}\right)\cos 2\pi X_0 f_0\right]$$

$$= \frac{-jA}{2}\exp\left[j\left(kz+\frac{\pi}{2}\right)\right]\left[1+m\exp\left(\frac{-j2\pi^2 z f_0^2}{k}\right)\cos 2\pi X_0 f_0\right]$$

b) A GENERAL EXPRESSION FOR AMPLITUDE MODULATION OF THIS TYPE IN THE TIME DOMAIN (DSB):

$$S_m(t) = A_1[1+m(t)]\cos\omega_0 t$$

$$\Rightarrow |S_m(t)|^2 = A_1^2[1+m(t)]^2\cos\omega_0 t$$

NOW

$$|U(X_0,Y_0)|^2 = \frac{A^2}{4}\left|1+m\exp\left(\frac{-j2\pi^2 z f_0^2}{k}\right)\cos 2\pi X_0 f_0\right|^2$$

$$= \frac{A^2}{4}\left|1+m\cos\frac{2\pi^2 z f_0^2}{k}\cos 2\pi X_0 f_0 - jm\sin\frac{2\pi^2 z f_0^2}{k}\cos 2\pi X_0 f_0\right|^2$$

$$= \frac{A^2}{4}\left[\left(1+m\cos\frac{2\pi^2 z f_0^2}{k}\cos 2\pi X_0 f_0\right)^2 + m^2\sin^2\left(\frac{2\pi^2 z f_0^2}{k}\right)\cos^2(2\pi X_0 f_0)\right]$$

$$= \frac{A^2}{4}\left[1+2m\cos\frac{2\pi^2 z f_0^2}{k}\cos 2\pi X_0 f_0 + m^2\cos^2(2\pi X_0 f_0)\right]$$

TO MAKE ANALOGY WITH EXPRESSION FOR $|S_m(t)|^2$;

$$\cos\frac{2\pi^2 z f_0^2}{k} = \pm 1$$

$$\Rightarrow \frac{2\pi^2 z f_0^2}{k} = n\pi \ni n = 0, \pm 1, \pm 2, \ldots$$

$$\therefore z = \frac{nk}{2\pi f_0^2} = \frac{n}{\lambda f_0^2}$$ FOR PURE AMPLITUDE MODULATION

AND NOW, AN EXCERCISE IN HAND-WAVING:

A GENERAL EXPRESSION FOR PHASE MODULATION:

$$s_m(t) = A_p \cos 2\pi [f_o' t + W_2 m(t)]$$

$$\Rightarrow |s_m(t)|^2 = A_p^2 \cos^2 2\pi [f_o' t + W_2 m(t)]$$

$$= \frac{A_p^2}{2}[1 + \cos 4\pi (f_o' t + W_2 m(t))]$$

AGAIN;

$$|U(x_o, y_o)|^2 = \frac{A^2}{4}[1 + 2m \cos \frac{2\pi^2 z f_o^2}{K} \cos 2\pi x_o f_o + m^2 \cos^2(2\pi f_o x_o)]$$

THERE EXISTS NO CORRELATION BETWEEN m AND $m(t)$, SINCE THE FORMER DETERMINES AMPLITUDE, AND THE LATTER PHASE.

FOR $m \ll 1$, $m^2 \lll 1$

$$\Rightarrow |U(x_o, y_o)|^2 \simeq \frac{A^2}{4}[1 + 2m \cos \frac{2\pi^2 z f_o^2}{K} \cos 2\pi x_o f_o]$$

$$= \frac{A^2}{4}[1 + m\{\cos 2\pi f_o\{\frac{\pi z}{K} f_o + x_o\} + \cos 2\pi f_o \{\frac{\pi z}{K} f_o - x_o\}\}]$$

$$= \frac{A^2}{2}[(1 + 2m \cos 2\pi f_o\{\frac{\pi z}{K} f_o + x_o\})]$$

$$+ \frac{A^2}{2}[1 + 2m \cos 2\pi f_o \{\frac{\pi z}{K} f_o - x_o\}]$$

THUS, $\frac{\pi f_o^2}{2K}$ MAY BE LOOKED UPON AS MODULATING x_o IN THE z DIRECTION. (WELL, MAYBE).

It's as good a guess as

I could make.

20/20

51-52 BOB MARKS

SPECIAL PROBLEM ON ANGULAR SPECTRUM

The transmittance function of an <u>infinite aperture</u> (some aperture!) is

$$t(x,y) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi x}{\lambda_0}\right)\cos\left(\frac{2\pi y}{\lambda_0}\right)\right].$$

The aperture is illuminated by a plane-wave of magnitude A propagating in the z-direction (the aperture plane is the x-y plane, z=0).

(a) Show that the field at z=0 has an angular spectrum

$$A(f_x,f_y) = \frac{A}{2}\left\{\delta(f_x)\delta(f_y) + \frac{1}{4}\left[\delta\left(f_x-\frac{1}{\lambda_0}\right)\delta\left(f_y-\frac{1}{\lambda_0}\right) + \delta\left(f_x-\frac{1}{\lambda_0}\right)\delta\left(f_y+\frac{1}{\lambda_0}\right) + \delta\left(f_x+\frac{1}{\lambda_0}\right)\delta\left(f_y-\frac{1}{\lambda_0}\right) + \delta\left(f_x+\frac{1}{\lambda_0}\right)\delta\left(f_y+\frac{1}{\lambda_0}\right)\right]\right\}.$$

(b) Show that the field at the plane z=d is given by

$$U(x,y,d) = \frac{A}{2}\left\{e^{j2\pi\frac{z}{\lambda}} + \frac{1}{2}e^{j2\pi z\left(\frac{1}{\lambda^2}-\frac{2}{\lambda_0^2}\right)^{1/2}}\left[\cos 2\pi\left(\frac{x+y}{\lambda_0}\right) + \cos 2\pi\left(\frac{x-y}{\lambda_0}\right)\right]\right\}.$$

Interpret each of these terms. Consider the cases:

(1) $\lambda < \lambda_0/\sqrt{2}$

(2) $\lambda > \lambda_0/\sqrt{2}$

a) FOR PLANE WAVE OF MAGNITUDE A @ $z=0$ $(U_i = A e^{jkz})$

$$A(f_X, f_Y) = \mathcal{F}^{+1}\{A\, t(X,Y)\}$$
$$= \mathcal{F}^{+1}\{\tfrac{A}{2}[1 + \cos(\tfrac{2\pi X}{\lambda_0})\cos(\tfrac{2\pi Y}{\lambda_0})]\}$$

NOW; $\mathcal{F}^{+1}\{\cos(\frac{2\pi X}{\lambda_0})\} = \frac{1}{2}[\delta(f_X - 1/\lambda_0) + \delta(f_X + 1/\lambda_0)]$

$$\Rightarrow A(f_X, f_Y) = \frac{A}{2}[\delta(f_X, f_Y) + \tfrac{1}{4}\{\delta(f_X - 1/\lambda_0) + \delta(f_X + 1/\lambda_0)\} \cdot \{\delta(f_Y - 1/\lambda_0) + \delta(f_Y + 1/\lambda_0)\}]$$
$$= \frac{A}{2}[\delta(f_X)\delta(f_Y) + \tfrac{1}{4}\{\delta(f_X - 1/\lambda_0)\delta(f_Y - 1/\lambda_0) + \delta(f_X - 1/\lambda_0)\delta(f_Y + 1/\lambda_0) + \delta(f_X + 1/\lambda_0)\delta(f_Y - 1/\lambda_0) + \delta(f_X + 1/\lambda_0)\delta(f_Y + 1/\lambda_0)\}]$$

b) $U(X,Y,Z) = \int_{-\infty}^{\infty}\int A(f_X, f_Y)\exp[j2\pi\{f_X X + f_Y Y + Z(\frac{1}{\lambda^2} - f_X^2 - f_Y^2)^{\frac{1}{2}}\}]df_X df_Y$

FROM THE DEL SIFTING PROPERTY, AND EXPRESSION FOR $A(f_X, f_Y)$ ABOVE:

$$U(X,Y,Z) = \frac{A}{2}[\exp(\tfrac{j2\pi Z}{\lambda}) + \tfrac{1}{4}\{\exp(j2\pi[\tfrac{X+Y}{\lambda_0} + Z(\tfrac{1}{\lambda^2} - \tfrac{2}{\lambda_0^2})^{\frac{1}{2}}]) + \exp(j2\pi[\tfrac{X-Y}{\lambda_0} + Z(\tfrac{1}{\lambda^2} - \tfrac{2}{\lambda_0^2})^{\frac{1}{2}}]) + \exp(j2\pi[\tfrac{-X+Y}{\lambda_0} + Z(\tfrac{1}{\lambda^2} - \tfrac{2}{\lambda_0^2})^{\frac{1}{2}}]) + \exp(j2\pi[\tfrac{-X-Y}{\lambda_0} + Z(\tfrac{1}{\lambda^2} - \tfrac{2}{\lambda_0^2})^{\frac{1}{2}}])\}]$$
$$= \frac{A}{2}[\exp(\tfrac{j2\pi Z}{\lambda}) + \tfrac{1}{4}\{\exp(j2\pi Z[\tfrac{1}{\lambda^2} - \tfrac{2}{\lambda_0^2}]^{\frac{1}{2}})[\exp\tfrac{j2\pi(X+Y)}{\lambda_0} + \exp\tfrac{-j2\pi(X+Y)}{\lambda_0} + \exp\tfrac{j2\pi(X-Y)}{\lambda_0} + \exp\tfrac{-j2\pi(X-Y)}{\lambda_0}]\}]$$
$$= \frac{A}{2}[e^{\frac{j2\pi Z}{\lambda}} + \tfrac{1}{2}e^{j2\pi Z(\frac{1}{\lambda^2} - \frac{2}{\lambda_0^2})^{1/2}}\{\cos\tfrac{2\pi(X+Y)}{\lambda_0} + \cos\tfrac{2\pi(X-Y)}{\lambda_0}\}]$$

FOR $\lambda > \lambda_0/\sqrt{2}$, $U(X,Y,Z)$ WILL ENTER THE EVANESCENT MODE, (*i.e.* THE $e^{j2\pi Z(\frac{1}{\lambda^2} - \frac{2}{\lambda_0^2})^{\frac{1}{2}}}$ BECOMES REAL INSTEAD OF COMPLEX), YIELDING AN EXPODENTIAL DECAY IN THE Z DIRECTION OF PART OF THE WAVE. THE LIMIT $(\lambda \to \infty)$ PRODUCES THE MAXIMUM DAMPING: $e^{-2\sqrt{2}\pi Z/\lambda_0}$. SO ROUGHLY, IF THE DIMENSION OF THE WAVE'S WAVELENGTH IS MUCH GREATER THAN THAT OF THE APERTURE, DAMPING WILL OCCUR, AND FURTHERMORE, AT LARGE

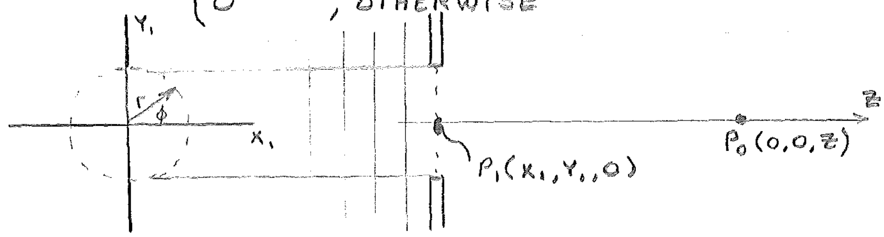
DISTANCES Z FROM THE APERTURE, THE $\frac{A}{2}e^{jkz}$ TERM OF $U(X,Y,Z)$ WILL DOMINATE. THUS, NO DIFFRACTION, ONLY A GLIMPSE OF ½ THE INCIDENT WAVE ($U_i = Ae^{jkz}$). DAMPING OF THIS SORT IS OF SMALL CONCERN AT OPTICAL FREQUENCIES.

This is precisely why you can't see detail less than a wavelength even at optical frequencies.

FOR $\lambda < \lambda_0/\sqrt{2}$, THE INCIDENT WAVE IS DIVIDED INTO TWO CAMPS, ONE BEING THE FOREMENTIONED REDUCED INCIDENT WAVE ($\frac{A}{2}e^{jkz}$). THE TERM $\cos\frac{2\pi(X+Y)}{\lambda_0} + \cos\frac{2\pi(X-Y)}{\lambda_0}$ ($= 2\cos\frac{2\pi}{\lambda_0}X \cos\frac{2\pi}{\lambda_0}Y$) IS A STANDING WAVE IN THE X·Y PLANE, WITH RECTANGULAR NODES @ $X=(n+1)\lambda_0, n=0,\pm1,\pm2,\ldots$; AND $Y=(m+1)\lambda_0, m=0,\pm1,\pm2,\ldots$, AND CHANGING PHASE IN ACCORDANCE TO THE $e^{j2\pi z\left(\frac{1}{\lambda^2}-\frac{1}{\lambda_0^2}\right)^{\frac{1}{2}}}$ TERM.

4·4) a) $t(x_1, y_1) = circ\sqrt{x_1^2+y_1^2}$

$$= \begin{cases} 1 & ; \sqrt{x_1^2+y_1^2} = r \le 1 \\ 0 & ; \text{OTHERWISE} \end{cases}$$



FRESNEL APPROXIMATION: (Pg 60)

$$U(P_0) = \frac{1}{j\lambda z} e^{jk[z+\frac{1}{2z}(x_0^2+y_0^2)]} \int_{-\infty}^{\infty}\int U(x_1,y_1)\, e^{j[k\frac{1}{2z}(x_1^2+y_1^2) - \frac{2\pi}{\lambda z}(x_0x_1+y_0y_1)]} dx_1 dy_1$$

ON APERTURE AXIS, $x_0 = y_0 = 0$

$$\Rightarrow U(P_0) = \frac{1}{j\lambda z} e^{jkz} \int_{-\infty}^{\infty}\int U(x_1,y_1)\, e^{j\frac{k}{2z}(x_1^2+y_1^2)} dx_1 dy_1$$

$$U_i = e^{jkz} \Rightarrow U(P_1) = e^{jkz}\, t(x_1,y_1)\big|_{z=0} = t(x_1,y_1) = circ\sqrt{x_1^2+y_1^2}$$

$$\Rightarrow U(P_0) = \frac{1}{j\lambda z} e^{jkz} \int\int_{\substack{\text{UNIT}\\ \text{CIRCLE}}} e^{\frac{jk}{2z}(x_1^2+y_1^2)}\, circ\sqrt{x_1^2+y_1^2}\, dx_1 dy_1$$

IN CYLIDRICAL COORDINATES: $circ\sqrt{x_1^2+y_1^2} = 1$; $r_1^2 = x_1^2+y_1^2$; $dx\,dy = r\,dr\,d\phi$

$$\Rightarrow U(P_0) = \frac{1}{j\lambda z} e^{jkz} \int_{\phi=0}^{2\pi}\int_{r=0}^{1} e^{\frac{jkr_1^2}{2z}} r_1\, dr_1\, d\phi$$

$$= \frac{1}{j\lambda z} e^{jkz} \int_{\phi=0}^{2\pi} \frac{z}{jk} e^{\frac{jkr_1^2}{2z}}\Big|_0^1 d\phi$$

$$= \frac{-1}{\lambda k} e^{jkz} \left(2\pi\{e^{jk/2z} - 1\}\right) \ni k = \frac{2\pi}{\lambda}$$

$$= -e^{jk(z+\frac{1}{2z})} + e^{jkz}$$

$$I(P_0) = [U U^*]$$

$$= \left[\left(-e^{jk(z+\frac{1}{2z})} + e^{jkz}\right)\left(-e^{-jk(z+\frac{1}{2z})} + e^{-jkz}\right)\right]$$

$$= \left[2 - e^{-jkz+jk(z+\frac{1}{2z})} - e^{jkz-jk(z+\frac{1}{2z})}\right]$$

$$= 2\left[1 - \cos\frac{k}{2z}\right]$$ (COSINE ARGUMENT IS DIMENSIONLESS BECAUSE K AND Z ARE NORMALIZED TO r WHICH WAS EQUATED TO UNITY)

b) $t(x_1, y_1) = 1 \qquad a \leq \sqrt{x_1^2 + y_1^2} \leq 1$ OR $a \leq r \leq 1$

$\qquad\qquad = 0$ OTHERWISE

FROM (a)

$$U(P_0) = \frac{1}{j\lambda z} e^{jkz} \int_{\phi=0}^{2\pi} \int_{r=a}^{1} e^{\frac{jkr_1^2}{2z}} r_1 \, dr_1 \, d\phi$$

$$= \frac{1}{j\lambda z} e^{jkz} \int_0^{2\pi} \frac{z}{jk} \left[e^{jk/2z} - e^{jka^2/2z} \right] d\phi$$

$$= -e^{jkz} \left[e^{jk/2z} - e^{jka^2/2z} \right]$$

$$= -e^{jk(z + 1/2z)} + e^{jk(z + \frac{a^2}{2z})}$$

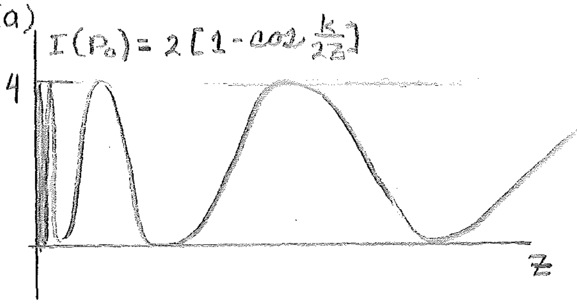
$$I(P_0) = U U^*$$

$$= \left(-e^{jk(z + 1/2z)} + e^{jk(z + \frac{a^2}{2z})} \right) \left(-e^{-jk(z + \frac{k}{2z})} + e^{-jk(z + \frac{a^2}{2z})} \right)$$

$$= 2 - e^{jk(\frac{1}{2z} - \frac{a^2}{2z})} - e^{-jk(\frac{1}{2z} - \frac{a^2}{2z})}$$

$$= 2\left[1 - \cos \frac{k}{2z}(1 - a^2) \right]$$

CASE (a)



MAXIMA @ $z = \frac{k}{2(2n+1)\pi}$

MINIMA @ $z = \frac{k}{4n\pi}$

$n = 0, 1, 2, \ldots$

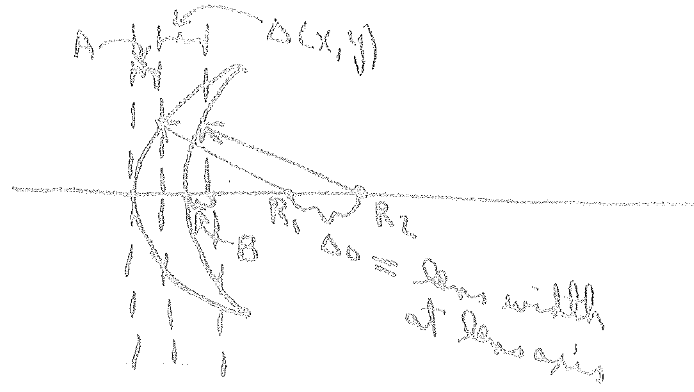
CASE (b) (SAME GENERAL GRAPH)

MAXIMA @ $z = \frac{k(1-a^2)}{2(2m+1)\pi}$

MINIMA @ $z = \frac{k(1-a^2)}{4m\pi}$

THE SAME GENERAL INTENSITY PATTERN OCCURS IN BOTH CASES, A $\sin \frac{1}{x}$ CURVE. INCREASING z DECREASES $\delta I / \delta z$, WHILE BETWEEN 0 AND Δz, THE INTENSITY CHANGES AN INFINITE NUMBER OF TIMES, FROM MAXIMUM TO MINIMUM. THE INTRODUCTION OF a DECREASED THE RESPECTIVE SPACINGS BETWEEN MAXIMA AND MINIMA (ie $I(P_0)_a|_{n+1} + I(P_0)_a|_n > I(P_0)_b|_{m+1} + I(P_0)_b|_m$), AND FURTHER INCREASING a WOULD PRODUCE SMALLER SPACINGS, THE LIMIT BEING $a \to 1$, IN WHICH CASE THE INTENSITY PATTERN WOULD BUNCH CLOSER & CLOSER TO THE ORIGIN

5-1 Positive meniscus lens



$$A = R_1 - \sqrt{R_1^2 - x^2 - y^2}$$

$$B = R_2 - \sqrt{R_2^2 - x^2 - y^2}$$

$$\Delta(x,y) = \Delta_0 - A + B$$

$$= \Delta_0 - \left(R_1 - \sqrt{R_1^2 - x^2 - y^2}\right) + \left(R_2 - \sqrt{R_2^2 - x^2 - y^2}\right)$$

$$= \Delta_0 - \left(R_1\left[1 - \sqrt{1 - \frac{(x^2+y^2)}{R_1^2}}\right]\right) + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

$$\cong \Delta_0 - \left(R_1\left[1 - \left(1 - \frac{x^2+y^2}{2R_1^2}\right)\right]\right) + R_2\left(1 - \left(1 - \frac{x^2+y^2}{2R_2^2}\right)\right)$$

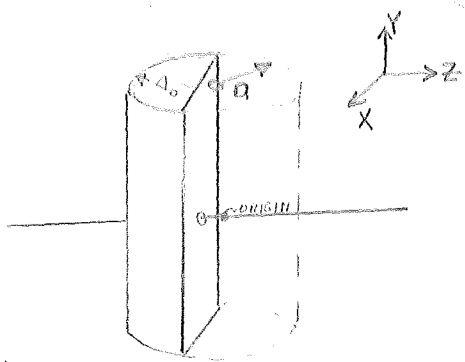
$$= \Delta_0 - R_1\frac{x^2+y^2}{2R_1^2} + R_2\frac{x^2+y^2}{2R_2^2} = \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

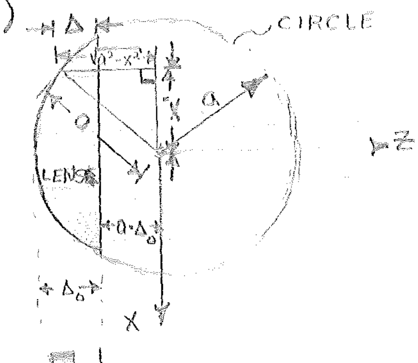
$$t_l(x,y) = \exp(jkn\Delta_0)\exp\left[-jk(n-1)\frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]$$

$$= \exp(jkn\Delta_0)\exp\left(-\frac{jk}{2f}\left[x^2+y^2\right]\right)$$

where $\frac{1}{f} \triangleq (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ and f is positive since $R_1 < R_2$.

10/10

5-2) 

a) 

FROM GEOMETRY:

$$\Delta(x,y)=\begin{cases}\sqrt{a^2-x^2}-(a-\Delta_0) & ; |x|\le\sqrt{a^2-(a-\Delta_0)^2} \\ 0 & ; \text{OTHERWISE}\end{cases}$$

$$=\Delta_0-a+\sqrt{a^2-x^2} \quad ; \text{(LIMITS UNDERSTOOD)}$$

$$=\Delta_0-a\left(1-\sqrt{1-\left(\tfrac{x}{a}\right)^2}\right)$$

PARAXIAL APPROXIMATION (IN ONE DIMENSION):

$$\sqrt{1-\left(\tfrac{x}{a}\right)^2}\simeq 1-\frac{x^2}{2a^2} \quad \text{FOR SMALL } \left(\tfrac{x}{a}\right)$$

THUS: $U(x,y)\cong\Delta_0-a\left[1-\left(1-\frac{x^2}{2a^2}\right)\right]$

$$=\Delta_0-\frac{x^2}{2a}$$

NOW: $t_\ell(x,y)=\exp(jk\Delta_0)\exp[jk(n-1)\Delta(x,y)]$

$$=\exp(jk\Delta_0)\exp\left[jk(n-1)\left(\Delta_0-\frac{x^2}{2a}\right)\right]$$

ALSO: $f\triangleq\frac{a}{n-1}$

$\Rightarrow t_\ell(x,y)=\exp(jk\Delta_0)\exp\left[jk\left\{(n-1)\Delta_0-\frac{x^2}{2f}\right\}\right]$

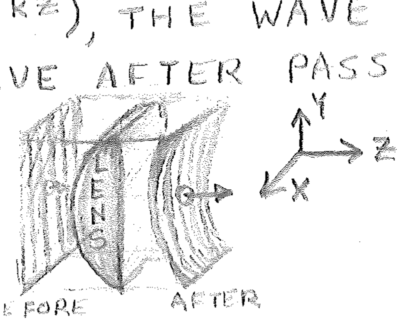
$$=\exp(jnk\Delta_0)\exp\left(\frac{-jkx^2}{2f}\right)$$

b) THE $\exp(jnk\Delta_0)$ TERM IS SIMPLY A CONSTANT PHASE DELAY, WHILE THE $\exp\left(\frac{-jkx^2}{2f}\right)$ TERM MAY BE CONSIDERED A QUADRADIC APPROXIMATION OF A CYLINDRICAL WAVE $\left(\frac{1}{x}\exp jkx\right)$. WITH AN INCIDENT PLANE WAVE (Ae^{jkz}), THE WAVE BECOMES A CYLINDRICAL WAVE AFTER PASSING THRU THE LENSE. (@ $z\simeq 0$)

$U_\ell'(x,y)=t(x,y)\,U(x,y)$

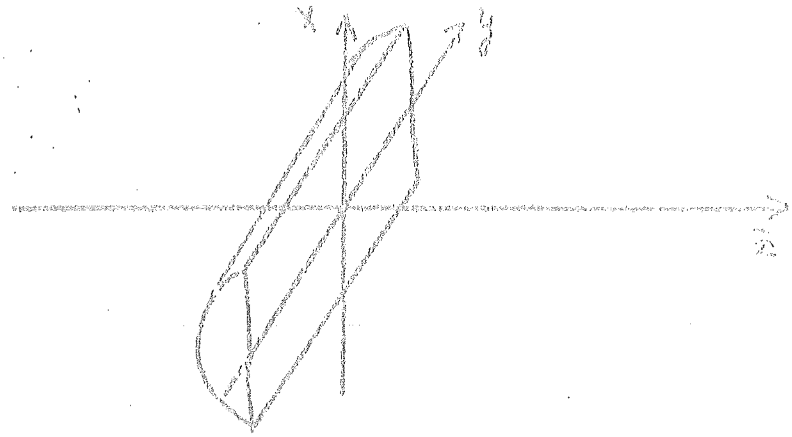
$\simeq Ae^{jnk\Delta_0}e^{-jkx^2/2f}$

(SMALL x/a)

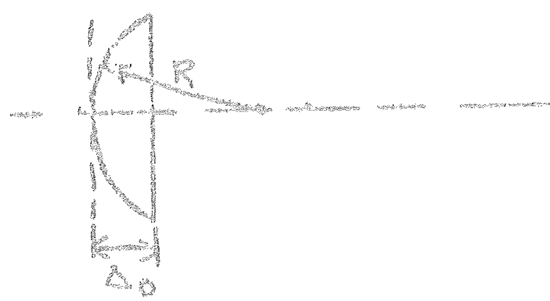


Solution to Prob. 5.2

(a)



Cross-section:



$$\Delta(x,y) = \Delta_0 - \left(R - \sqrt{R^2 - x^2}\right) = \Delta_0 - R + \sqrt{R^2 - x^2} \quad \text{independent of } y$$

$$= \Delta_0 - R\left(1 - \sqrt{1 - \frac{x^2}{R^2}}\right)$$

$$\cong \Delta_0 - R\left[1 - \left(1 - \frac{x^2}{2R^2}\right)\right] = \Delta_0 - \frac{x^2}{2R}$$

$$t_A(x,y) = \exp[jk\Delta_0] \exp\left[jk(n-1)\left(\Delta_0 - \frac{x^2}{2R}\right)\right]$$

$$= \exp[jkn\Delta_0] \exp\left[-jk(n-1)\frac{x^2}{2R}\right]$$

$$= \exp[jkn\Delta_0] \exp\left[-j\frac{k}{2f}x^2\right] \quad \text{where } f \triangleq \frac{R}{(n-1)}$$

since $R_2 = \infty$

(b) An incident plane wave is transformed to a quadratic approximation to a cylindrical wave which is focused to a line at f parallel to the y-axis.

BOB MARKS

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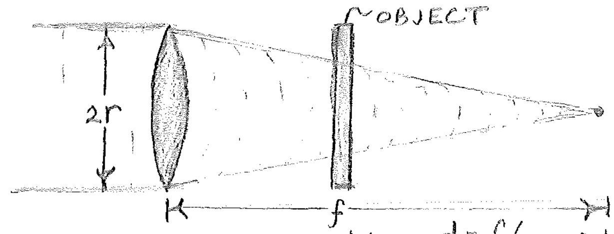
5-8)



$$t_0(x_0, y_0) = \frac{1}{2}\left[1 + \cos(2\pi f_0 x_0)\right] \mathrm{rect}\left(\frac{x_0}{L}\right) \mathrm{rect}\left(\frac{y_0}{L}\right)$$

AND:

$$U_f(x_f, y_f) = \frac{Af}{d}\,\frac{1}{j\lambda d} \exp\left[\frac{jk}{2d}(x_f^2 + y_f^2)\right] \int\!\!\int_{-\infty}^{\infty} t_0(x_0, y_0)\, P\left(\frac{x_0 f}{d}, \frac{y_0 f}{d}\right) \cdot \exp\left[\frac{-j2\pi}{\lambda d}(x_0 x_f + y_0 y_f)\right] dx_0\, dy_0$$

FOR A CIRCULAR LENS

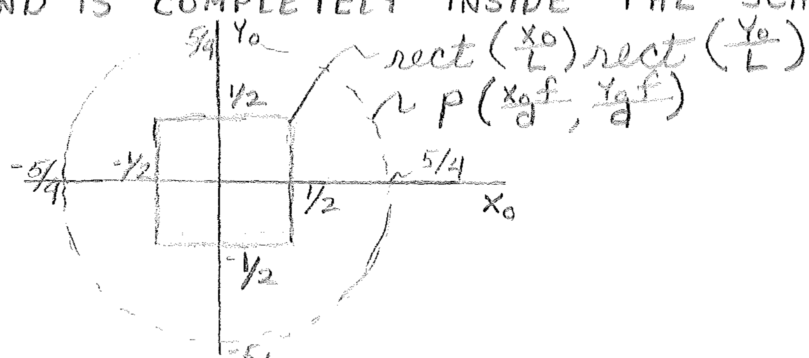
$$P(x_0, y_0) = \mathrm{circ}\left[\frac{1}{r}\sqrt{x_0^2 + y_0^2}\right]$$

$$\Rightarrow P\left(\frac{x_0 f}{d}, \frac{x_0 f}{d}\right) = \mathrm{circ}\left[\frac{f}{dr}\sqrt{x_0^2 + y_0^2}\right]$$

NOW $f = 200\,\mathrm{cm}$, $d = 100\,\mathrm{cm}$, AND $r = 5/2\ \mathrm{cm}$

$$\Rightarrow P\left(\frac{x_0 f}{d}, \frac{y_0 f}{d}\right) = P(2x_0, 2y_0) = \mathrm{circ}\left[\frac{4}{5}\sqrt{x_0^2 + y_0^2}\right].$$

THE PUPIL FUNCTION ON THE OBJECT PLANE IS THUS A CIRCLE OF RADIUS $\frac{5}{4}$ cm, AN OBVIOUS CONCLUSION, IN THAT THE DIAMETER OF THE DIVERGING RAY IS ONE HALF OF THE ORIGINAL PUPIL FUNCTION @ f/2. THE $\mathrm{rect}\left(\frac{x_0}{L}\right)\mathrm{rect}\left(\frac{y_0}{L}\right)$ TERM IN $t_0(x, y)$ DEFINES A UNIT SQUARE CENTERED AT THE ORIGIN $(L = 1)$, AND IS COMPLETELY INSIDE THE SCALED PUPIL FUNCTION:



THUS: $\mathrm{rect}\left(\frac{x_0}{L}\right)\mathrm{rect}\left(\frac{y_0}{L}\right) P\left(\frac{x_0 f}{d}, \frac{y_0 f}{d}\right) = \mathrm{rect}\left(\frac{x_0}{L}\right)\mathrm{rect}\left(\frac{y_0}{L}\right)$

$$\Rightarrow U_f(x_f,Y_f)=\frac{Af}{j\lambda d^2}\exp\left[\frac{jk}{2d}(x_f^2+Y_f^2)\right]\int_{-\infty}^{\infty}\!\!\int t_o(x_o,Y_o)\exp\left\{\frac{-j2\pi}{\lambda d}(x_o x_f+Y_oY_f)\right\}dx_o\,dY_o$$

$$=\frac{Af}{j\lambda d^2}\exp\left[\frac{jk}{2d}(x_f^2+Y_f^2)\right]\mathcal{F}\{t_o(x_o,Y_o)\}\Big|_{f_Y=Y_f/\lambda d,\ f_X=x_f/\lambda d}$$

$$\mathcal{F}\{t_o(x_o,Y_o)\}=\frac{1}{2}\mathcal{F}\left[(1+\cos 2\pi f_o x_o)\,\mathrm{rect}\left(\frac{x_o}{L}\right)\right]\mathcal{F}\left[\mathrm{rect}\left(\frac{Y_o}{L}\right)\right]$$ (SEPARABLE FUNCTIONS)

$$\mathcal{F}\left[(1+\cos 2\pi f_o x_o)\,\mathrm{rect}\left(\frac{x_o}{L}\right)\right]=\mathcal{F}\{1+\cos 2\pi f_o x_o\}*\mathcal{F}\left\{\mathrm{rect}\left(\frac{x_o}{L}\right)\right\}$$

$$=\left[\delta(f_x)+\tfrac{1}{2}\delta(f_x-f_o)+\tfrac{1}{2}\delta(f_x+f_o)\right]*L\,\mathrm{sinc}(f_xL)$$

$$=L\int_{-\infty}^{\infty}\left[\delta(\xi)+\tfrac{1}{2}\delta(\xi-f_o)+\tfrac{1}{2}\delta(\xi+f_o)\right]\mathrm{sinc}[(\xi-f_x)L]\,d\xi$$

$$=L\left[\mathrm{sinc}(-Lf_x)+\tfrac{1}{2}\mathrm{sinc}[(f_o-f_x)L]+\tfrac{1}{2}\mathrm{sinc}[-(f_o+f_x)L]\right]$$

$$=L\left[\mathrm{sinc}(Lf_x)+\tfrac{1}{2}\mathrm{sinc}[(f_x-f_o)L]+\tfrac{1}{2}\mathrm{sinc}[(f_o+f_x)L]\right]$$

BECAUSE $\mathrm{sinc}(\psi)=\mathrm{sinc}(-\psi)$

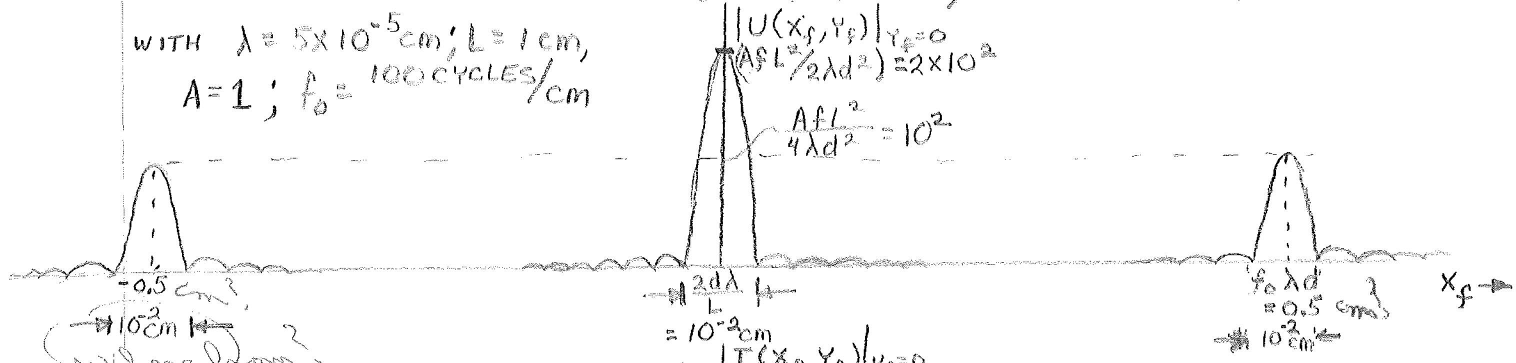
HENCE:

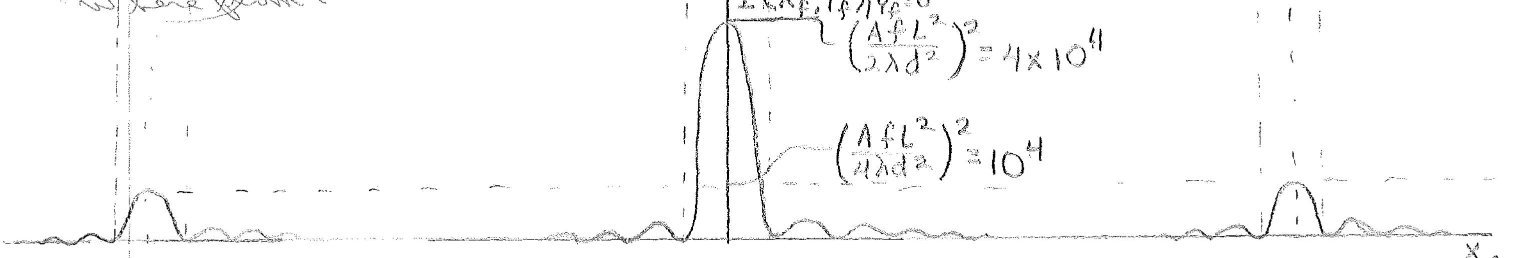
$$U_f(x_f,Y_f)=\frac{AfL^2}{j2\lambda d^2}\exp\left[\frac{jk}{2d}(x_f^2+Y_f^2)\right]\left[\mathrm{sinc}\left(\frac{Lx_f}{\lambda d}\right)+\frac{1}{2}\mathrm{sinc}\left(L\left\{\frac{x_f}{\lambda d}-f_o\right\}\right)+\frac{1}{2}\mathrm{sinc}\left(L\left\{\frac{x_f}{\lambda d}+f_o\right\}\right)\right]\mathrm{sinc}\left(\frac{LY_f}{\lambda d}\right)$$

$$I(x_f,Y_f)=|U(x_f,Y_f)|^2$$

$$=\left(\frac{AfL^2}{2\lambda d^2}\right)^2\left[\mathrm{sinc}\left(\frac{Lx_f}{\lambda d}\right)+\frac{1}{2}\mathrm{sinc}\left\{L\left(\frac{x_f}{\lambda d}-f_o\right)\right\}+\frac{1}{2}\mathrm{sinc}\left\{L\left\{\frac{x_f}{\lambda d}-f_o\right\}\right\}\right]^2\mathrm{sinc}^2\frac{LY_f}{\lambda d}$$

WITH $\lambda=5\times10^{-5}$ cm; $L=1$ cm,
$A=1$; $f_o=100$ CYCLES/cm





DUE TO THE PULSES NEAR ORTHOGONALITY:

$$I(x_f,Y_f)|_{Y_f=0}\doteq\left(\frac{AfL^2}{2\lambda d^2}\right)^2\left[\mathrm{sinc}^2\left(\frac{Lx_f}{\lambda d}\right)+\frac{1}{4}\mathrm{sinc}^2\left\{L\left(\frac{x_f}{\lambda d}-f_o\right)\right\}+\frac{1}{4}\mathrm{sinc}^2\left\{L\left(\frac{x_f}{\lambda d}+f_o\right)\right\}\right]$$

THIS DIFFRACTION PROCESS IS ANALAGOUS TO AM HETRODYNING OF BASEBAND SIGNAL $\mathrm{sinc}(Lx_f/\lambda d)$ TO CARRIER FREQUNCY $x_{fc}(=f_o\lambda d)$, THE SIGNAL THUS BEING THE "FRAME" OF $t(x_o,Y_o)$.

$$t(x_0, y_0) = \tfrac{1}{2}(1 + \cos 2\pi f_x x_0)$$

and $f_0 = 100 \frac{\text{cycles}}{\text{cm}} = 10^4 \frac{\text{cycles}}{\text{m}}$.

$$\mathcal{F}\{t(x_0,y_0)\} = \left[\tfrac{1}{2}\delta(f_x, f_y) + \tfrac{1}{4}\delta(f_x + 10^4, f_y) + \tfrac{1}{4}\delta(f_x - 10^4, f_y)\right]$$

$$* \operatorname{sinc}(L f_x)\operatorname{sinc}(L f_y)$$

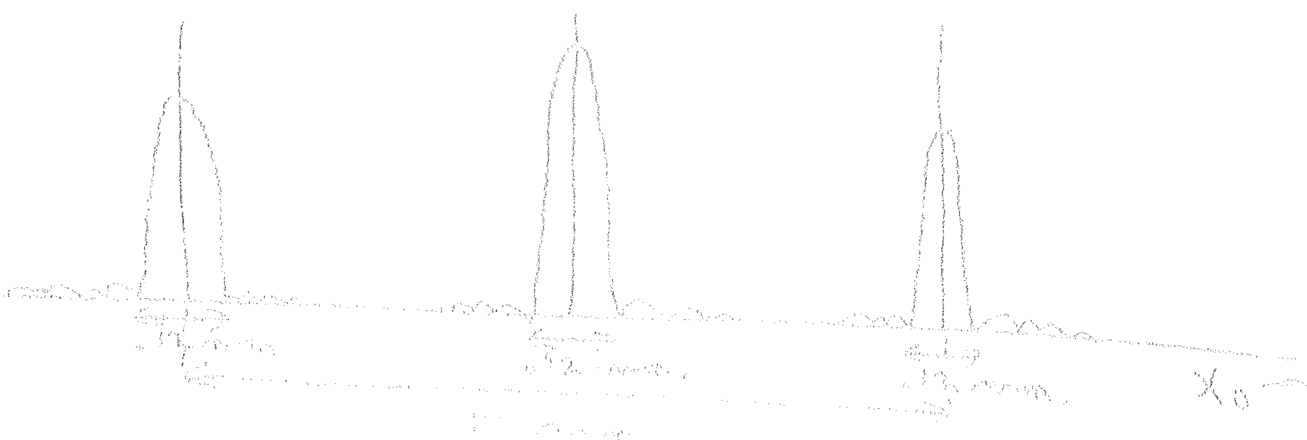
The frequency shift $f_x' = 10^4 \frac{\text{cycles}}{\text{m}} = \frac{x_0'}{\lambda d}$, & $\lambda = 6\times10^{-7}$ m.

$\therefore$ The sidebands are shifted by $x_0' = f_x' \lambda d = 10^4 \times 6\times10^{-7} \times 1$

$= \underline{6 \text{ mm}}$.

Also, $\operatorname{sinc}(L f_x) = \dfrac{\sin\left(\frac{\pi L x_0''}{\lambda d}\right)}{\frac{\pi L x_0''}{\lambda d}}$; the 1st zero occurs

when $\frac{\pi L x_0''}{\lambda d} = \pi$, or $x_0'' = \frac{6\times10^{-7}\times 1}{10^{-2}} = \underline{.06 \text{ mm}}$.



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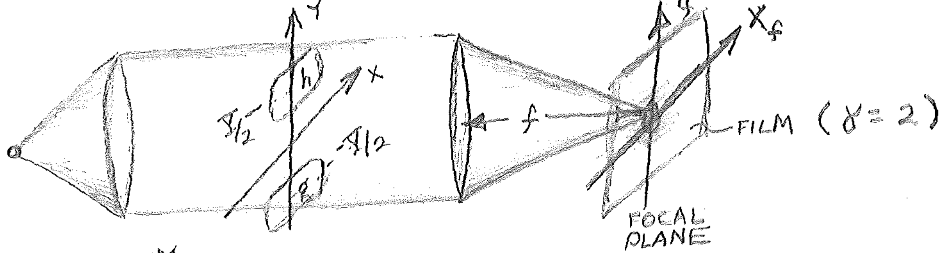
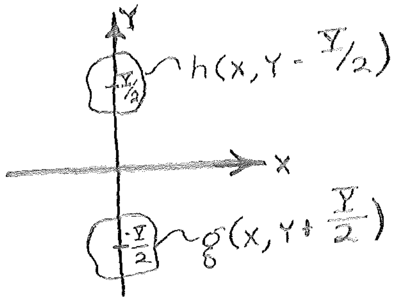
7-9)





IN THE FOCAL PLANE: $U_f(x_f, y_f) = \frac{1}{j\lambda f}\mathcal{F}\{t(x,y)\}\Big|_{\substack{f_y = y_f/\lambda f \\ f_x = x_f/\lambda f}}$ (UNIT AMPLITUDE PLANE WAVE)

WHERE $t(x,y) = h(x, y - \frac{Y}{2}) + g(x, y + \frac{Y}{2})$

$\Rightarrow U_f(x_f, y_f) = \frac{1}{j\lambda f}\mathcal{F}\{h(x, y - \frac{Y}{2}) + g(x, y + \frac{Y}{2})\}\Big|_{\substack{f_y = y_f/\lambda f \\ f_x = x_f/\lambda f}}$

FROM THE SHIFT THEM.: $\left(\mathcal{F}\{g(x-a)\} = G(f_x)e^{-j2\pi f_x a}\right)$ AND LIN. THEM

$U_f(x_f, y_f) = \frac{1}{j\lambda f}\left[H(f_x, f_y)e^{-j\pi f_y Y} + G(f_x, f_y)e^{j\pi f_y Y}\right]\Big|_{\substack{f_y = y_f/\lambda f \\ f_x = x_f/\lambda f}}$

ASSUMING POSITIVE TRANSPARENCY, THE RECORDED AMPLITUDE TRANSMITTANCE IS EXPRESSED AS:

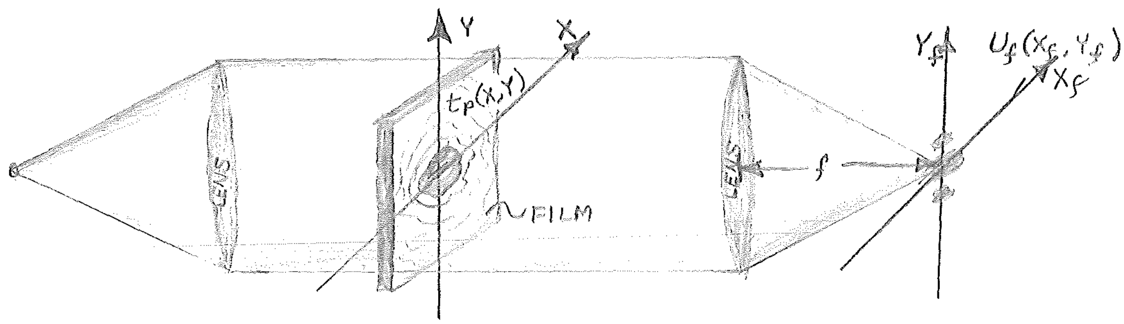
$$t_p(x_f, y_f) = k_p\,|U_f(x_f, y_f)|^{\gamma}$$
$$= k_p\,|U_f(x_f, y_f)|^2$$
$$= k_p\,U_f(x_f, y_f)\,U_f^*(x_f, y_f)$$

NOTING: $ab = c \Rightarrow a^* b^* = c^*$ AND $a + b = c \Rightarrow a^* + b^* = c^*$

$$t_p = \frac{k_p}{(\lambda f)^2}\left[H(f_x, f_y)e^{-j\pi f_y Y} + G(f_x, f_y)e^{j\pi f_y Y}\right]\left[H^*(f_x, f_y)e^{j\pi f_y Y} + G^*(f_x, f_y)e^{-j\pi f_y Y}\right]$$
$$= \frac{k_p}{(\lambda f)^2}\left[|H(f_x, f_y)|^2 + |G(f_x, f_y)|^2 + H(f_x, f_y)G^*(f_x, f_y)e^{-j2\pi f_y Y} + H^*(f_x, f_y)G(f_x, f_y)e^{j2\pi \ldots}\right.$$

t_p IS PURE REAL, DESPITE THE APPARENT PHASE FACTORS, IN THAT aa^* IS ALWAYS PURE REAL.

AGAIN: $f_x = x_f/\lambda f$ AND $f_y = x_y/\lambda f$.



$$t_p = \frac{K_p}{(\lambda f)^2}\left[\left|H\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)\right|^2 + \left|G\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)\right|^2 + H\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)G^*\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{-\frac{j2\pi\bar{Y}y}{\lambda f}} + H^*\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)G\left(\tfrac{x}{\lambda f},\tfrac{y}{f\lambda}\right)e^{\frac{j2\pi\bar{Y}y}{\lambda f}}\right]$$

AS BEFORE: $U_f(x_f, y_f) = \frac{1}{j\lambda f}\,\mathfrak{F}\{t_p(x,y)\}\Big|_{f_y = y_f/\lambda f,\ f_x = x_f/\lambda f}$

$$U_f(x_f,y_f) = \frac{1}{j\lambda f}\frac{K_p}{(\lambda f)^2}\Big[\mathfrak{F}\left\{H\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)H^*\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)\right\} + \mathfrak{F}\left\{G\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)G^*\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)\right\}$$
$$+ \mathfrak{F}\left\{H\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{-\frac{j\pi\bar{Y}y}{\lambda f}}G^*\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{-\frac{j\pi\bar{Y}y}{\lambda f}}\right\}$$
$$+ \mathfrak{F}\left\{H^*\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{\frac{j\pi\bar{Y}y}{\lambda f}}G\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{\frac{j\pi\bar{Y}y}{\lambda f}}\right\}\Big]$$

NOW, BOTH $h(x,y)$ AND $g(x,y)$ ARE PURE REAL, THUS THEIR TRANSFORM'S magnitude AND THE TRANSFORMS' CONJUGATES magnitudes ARE EVEN FUNCTIONS $\left(\left|H\left(\tfrac{x}{f\lambda},\tfrac{y}{f\lambda}\right)\right| = \left|H\left(\tfrac{-x}{f\lambda},\tfrac{-y}{f\lambda}\right)\right|\right.$, *etc.*$\left.\right)$

NOTE ALSO $a(x,y) * b^*(-x,-y) = a(x,y) \star b(x,y)$.

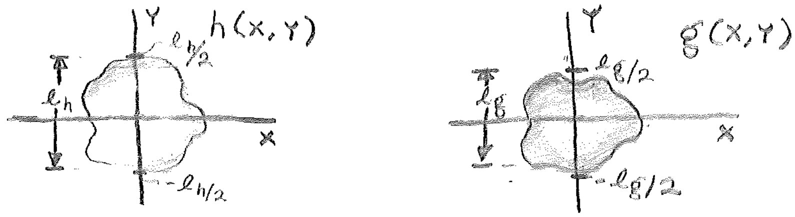
THE FACTORS $e^{\pm\frac{j\pi\bar{Y}y}{\lambda f}}$ MERELY REPRESENT SHIFTS OF $\pm\frac{\bar{Y}}{\lambda f}$ IN THE TRANSFORM PLANE.

the phase portions of the transforms are odd functions

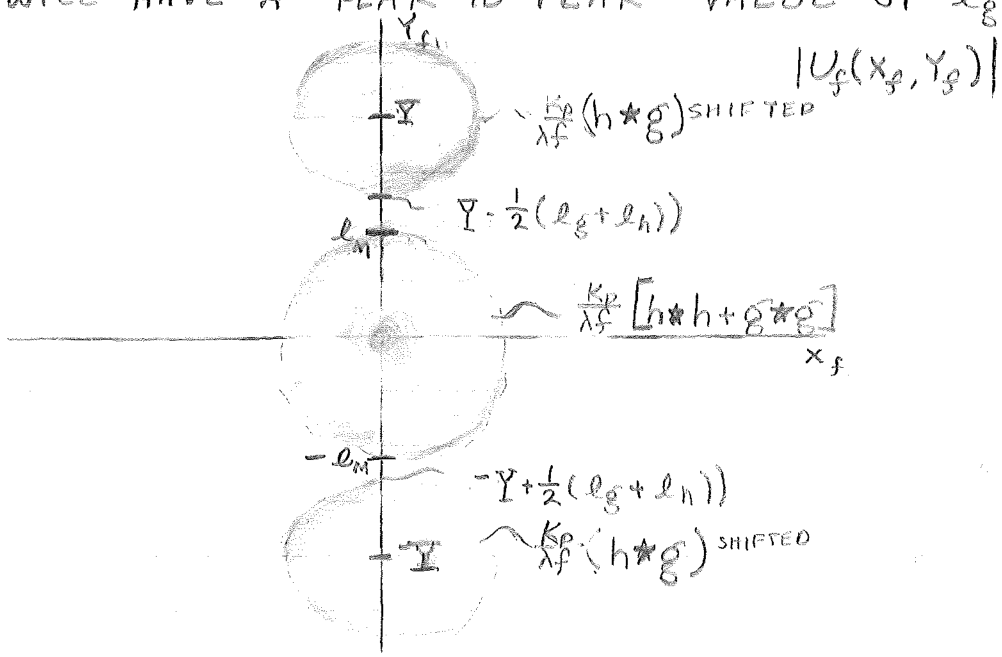
$$\Rightarrow U(x_f,y_f) = \frac{1}{j\lambda f}\frac{K_p}{(\lambda f)^2}\Big[\mathfrak{F}\left\{H\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)H^*\left(\tfrac{+x}{\lambda f},\tfrac{+y}{\lambda f}\right)\right\} + \mathfrak{F}\left\{G\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)G^*\left(\tfrac{+x}{\lambda f},\tfrac{+y}{\lambda f}\right)\right\}$$
$$+ \mathfrak{F}\left\{H\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{-\frac{j\pi\bar{Y}y}{\lambda f}}G^*\left(\tfrac{+x}{\lambda f},\tfrac{+y}{\lambda f}\right)e^{-\frac{j\pi\bar{Y}y}{\lambda f}}\right\}$$
$$+ \mathfrak{F}\left\{G\left(\tfrac{x}{\lambda f},\tfrac{y}{\lambda f}\right)e^{\frac{j\pi\bar{Y}y}{\lambda f}}H^*\left(\tfrac{+x}{\lambda f},\tfrac{+y}{\lambda f}\right)e^{j\pi\bar{Y}y/\lambda f}\right\}$$
$$= \frac{1}{j\lambda f}\frac{K_p}{(\lambda f)^2}\Big[(\lambda f)^2 h(\lambda f f_x, \lambda f f_y) \star h^*(+\lambda f f_x, +\lambda f f_y)$$
$$+ (\lambda f)^2 g(\lambda f f_x, \lambda f f_y) \star g(+\lambda f f_x, +\lambda f f_y)$$
$$+ (\lambda f)^2 h\left(\lambda f f_x, \lambda f\left(f_y - \tfrac{\bar{Y}}{2\lambda f}\right)\right) \star g\left(\lambda f f_x, \lambda f\left\{f_y - \tfrac{\bar{Y}}{2\lambda f}\right\}\right)$$
$$+ (\lambda f)^2 g\left(\lambda f f_x, \lambda f\left\{f_y + \tfrac{\bar{Y}}{2\lambda f}\right\}\right) \star h\left(\lambda f f_x, \lambda f\left\{f_y + \tfrac{\bar{Y}}{2\lambda f}\right\}\right)$$

NOW $f_x = \frac{x_f}{\lambda f}$; $f_y = \frac{y_f}{\lambda f}$

$$\Rightarrow U(x_f,y_f) = \frac{K_p}{j\lambda f}\Big[h(x_f,y_f)\star h(x_f,y_f) + g(x_f,y_f)\star g(x_f,y_f)$$
$$+ h\left(x_f, y_f - \tfrac{\bar{Y}}{2}\right)\star g\left(x_f, y_f - \tfrac{\bar{Y}}{2}\right)$$
$$+ h\left(x_f, y_f + \tfrac{\bar{Y}}{2}\right)\star g\left(x_f, y_f + \tfrac{\bar{Y}}{2}\right)\Big]$$



AS SHOWN, $h(x,y)$ HAS A "PEAK TO PEAK" PUPIL DIAMETER OF ℓ_h IN THE Y DIRECTION, AND $g(x,y)$ A SIMILAR VALUE OF ℓ_g. THE Y PUPIL OF $g * g$ WILL THUS HAVE A VALUE OF $2\ell_g$, AND $h * h$ OF $2\ell_h$. $h * g$ WILL HAVE A "PEAK TO PEAK" VALUE OF $\ell_g + \ell_h$.



$$\ell_M = \begin{cases} \ell_g & \text{IF } \ell_g \geq \ell_h \\ \ell_h & \text{IF } \ell_g < \ell_h \end{cases} \quad \text{ie } \ell_m = \text{MAXIMUM}(\ell_g, \ell_h)$$

THE CROSS-CORRELATION $\left(\frac{K_p}{\lambda f} g * h\right)$ CAN BE SEPARATED FROM THE OTHER COMPONENTS IF THERE IS NO OVERLAP.

$$\Rightarrow \quad \bar{Y} - \frac{1}{2}(\ell_g + \ell_h)) \geq \ell_m$$

$$\therefore \bar{Y} \geq \ell_m + \frac{1}{2}(\ell_g + \ell_h)$$

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Prob. 7-9 Solution

The total amplitude in the focal plane, from the superposition theorem, is the sum of the Fourier transform of g and h shifted.

$$\mathcal{F}\{g(x, y - \tfrac{Y}{2})\} = \frac{1}{\lambda f} G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \exp\left(-i \frac{Y}{2} \frac{y}{\lambda f}\right)$$

$$\mathcal{F}\{h(x, y + \tfrac{Y}{2})\} = \frac{1}{\lambda f} H\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \exp\left(i \frac{Y}{2} \frac{y}{\lambda f}\right)$$

$$I(x_f, y_f) = \left|\frac{1}{\lambda f} G(x_f, y_f) \exp\left(-i \frac{Y}{2} y_f\right) + \frac{1}{\lambda f} H(x_f, y_f) \exp\left(i \frac{Y}{2} y_f\right)\right|^2$$

$$= \left[\frac{1}{\lambda f} G(x_f, y_f) \exp\left(-i \frac{Y}{2} y_f\right) + \frac{1}{\lambda f} H(x_f, y_f) \exp\left(i \frac{Y}{2} y_f\right)\right]$$

$$\left[\frac{1}{\lambda f} G^*(x_f, y_f) \exp\left(i \frac{Y}{2} y_f\right) + \frac{1}{\lambda f} H^*(x_f, y_f) \exp\left(-i \frac{Y}{2} y_f\right)\right].$$

$\therefore$

$$I(x_f, y_f) = \frac{1}{\lambda^2 f^2}\left[\left|G(x_f, y_f)\right|^2 + \left|H(x_f, y_f)\right|^2\right.$$

$$+ G(x_f, y_f) H^*(x_f, y_f) \exp(-i Y y_f)$$

$$\left. + H(x_f, y_f) G^*(x_f, y_f) \exp(i Y y_f)\right]$$

$u_1 = G$

The amplitude transmittance of the exposed film is proportional to the intensity $I(x_f, y_f)$ since $\gamma = 2$

$$\therefore t_p = k_p I^{\gamma_p/2} = k_p I$$

Thus the input to the system during the second phase is $t(x,y) = k_p\, I(x,y)$ as given previously.
Then the amplitude in the focal plane is:

$$U(x_f, y_f) = \frac{k_p}{(\lambda f)^2}\Big[\mathcal{F}\{|G|^2\} + \mathcal{F}\{G H^* \exp(-i Y \tfrac{y}{\lambda f})\} + \mathcal{F}\{H G^* \exp(i Y \tfrac{y}{\lambda f})\} + \mathcal{F}\{|H|^2\}\Big]$$

$$= \frac{k_p}{(\lambda f)^2}\Big[g(x,y) * g^*(-x,-y) + g(x,y) * h^*(-x,-y) * \delta(x, y-Y) + h(x,y) * g^*(-x,-y) * \delta(x, y+Y) + h(x,y) * h^*(-x,-y)\Big]$$

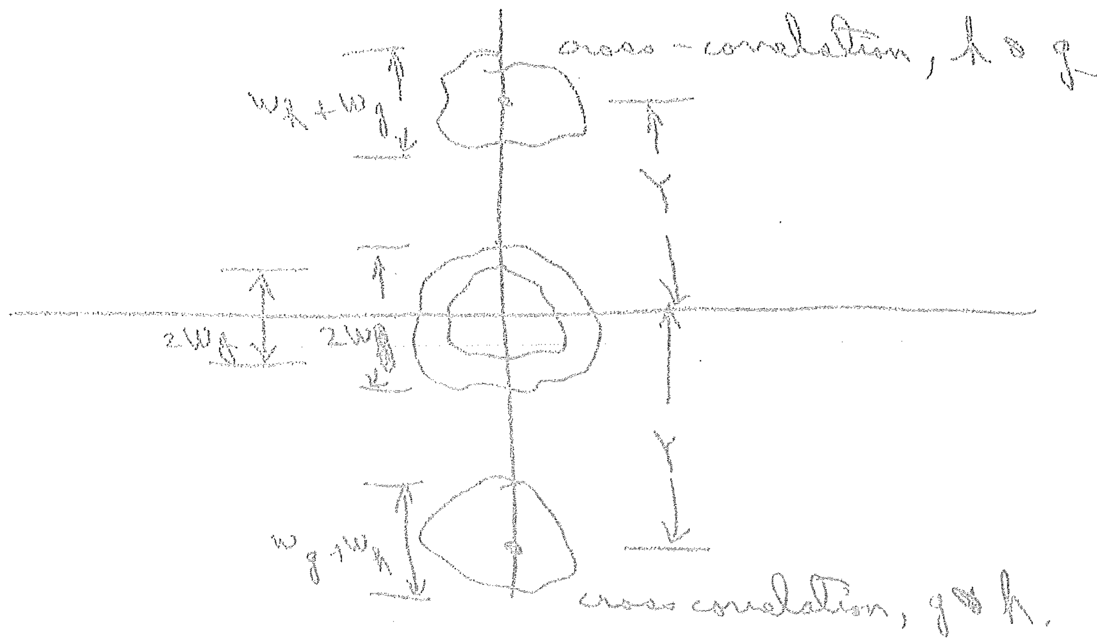
Let W_h and W_g be the max. widths of h and g, respectively. Then

a) $\frac{k_p}{(\lambda f)^2}\, g(x,y) * g^*(-x,-y) \rightarrow 2W_g$ width

b) $\frac{k_p}{(\lambda f)^2}\, g(x,y) * h^*(-x,-y) * \delta(x, y-Y) \Rightarrow W_g + W_h$

c) $\frac{k_p}{(\lambda f)^2}\, g^*(-x,-y) * h(x,y) * \delta(x, y+Y) \Rightarrow W_g + W_h$

d) $h(x,y) * h^*(-x,-y)\, \frac{k_p}{(\lambda f)^2} \Rightarrow 2W_h$.



Assuming $W_h > W_g$
complete separation of the cross correlation components will be realized if:

$$Y > W_h + \frac{W_h + W_g}{2}$$

or $\quad Y > \frac{3W_h}{2} + \frac{W_g}{2}$ when $W_h > W_g$.

BOB MARKS

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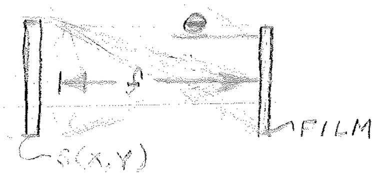
7.13)



FOR AN OBJECT PLACED AGAINST THE LENS: (Pg 85, EQ 5.15)

$$U_s(x_f,y_f) = \frac{A}{j\lambda f} \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} \int_{-\infty}^{\infty}\int s(x,y) \exp\left\{\frac{-j2\pi}{\lambda f}(x_f x + y_f y)\right\} dx\,dy$$

$$= \frac{A}{j\lambda f} \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} \mathcal{F}\left[s(x,y)\right]_{f_x = x_f/\lambda f}^{f_y = y_f/\lambda f}$$

$$= \frac{A}{j\lambda f} \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)$$

THE PRISM CONTRIBUTES THE FIELD DISTRIBUTION:

$$U_r(x_f,y_f) = r_0 \exp(-j2\pi\alpha y_f) \ni \alpha = \frac{\sin\theta}{\lambda}$$

THE TOTAL FIELD DISTRIBUTION AT THE FILM IS

$$U_f(x_f,y_f) = U_s(x_f,y_f) + U_r(x_f,y_f)$$

$$= \frac{A}{j\lambda f} \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) + r_0 \exp(-j2\pi\alpha y_f)$$

ASSUMING A γ OF 2, THE FILM WILL RECORD:

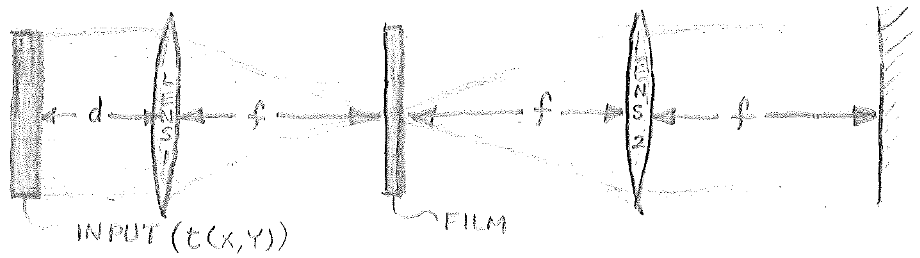
$$I(x_f,y_f) = k_p |U_f(x_f,y_f)|^2$$

$$= k_p (U_f(x_f,y_f))(U^*(x_f,y_f))$$

$$= k_p \left[\frac{A}{j\lambda f} \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) + r_0 \exp(-j2\pi\alpha y_f)\right]$$
$$\left[\frac{-A}{j\lambda f} \exp\left\{\frac{-jk}{2f}(x_f^2+y_f^2)\right\} S^*\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) + r_0 \exp(j2\pi\alpha y_f)\right]$$

$$= k_p \left[\left(\frac{A}{\lambda f}\right)^2 \left|S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)\right|^2 + r_0^2 + \frac{Ar_0}{j\lambda f} S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) \exp(j2\pi\alpha y_f) \cdot \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} - \frac{Ar_0}{j\lambda f} S^*\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) \exp(-j2\pi\alpha y_f) \exp\left\{\frac{-jk}{2f}(x_f^2+y_f^2)\right\}\right]$$

$$= k_p \left[\left(\frac{A}{\lambda f}\right)^2 \left|S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)\right|^2 + r_0^2 + \frac{Ar_0}{j\lambda f}\left\{S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) \exp(j2\pi\alpha y_f) \exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} - S^*\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right) \exp(-j2\pi\alpha y_f) \exp\left\{\frac{-jk}{2f}(x_f^2+y_f^2)\right\}\right\}\right]$$



THE INPUT IS TRANSFORMED TO $U_f(X,Y)$, WHERE (pg 81, 5.20

$$U_f(x_f, y_f) = \frac{A}{j\lambda f} \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right] \iint_{-\infty}^{\infty} t(x,y) \exp\left[\frac{-j2\pi}{\lambda f}(x_0 x_f + y_0 y_f)\right] dx\,dy$$

$$= \frac{A}{j\lambda f} \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right] T\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)$$

(PUPIL ASSUMED BIG ENOUGH)

MULTIPLYING BY THE PREVIOUSLY COMPUTED FILTER:

$$U(x_f, y_f) = U_f(x_f, y_f)\, I(x_f, y_f)$$

$$= \frac{A k_p}{j\lambda f} \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right] T\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)$$

$$\left[\left(\frac{A}{\lambda f}\right)^2 \left|S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)\right|^2 + r_0^2 + \frac{A r_0}{j\lambda f}\left\{S\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)\exp(j2\pi\alpha y_f)\right.\right.$$

$$\left.\left.\exp\left\{\frac{jk}{2f}(x_f^2+y_f^2)\right\} - S^*\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}\right)\exp(-j2\pi\alpha y_f)\exp\left\{\frac{-jk}{2f}(x_f^2+y_f^2)\right\}\right\}\right]$$

$$= \left(\frac{A}{\lambda f}\right)^3 \frac{k_p}{j} T|S|^2 \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right]$$

$$+ \frac{r_0^2 A k_p}{j\lambda f} T \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right]$$

$$- \left(\frac{A}{\lambda f}\right)^2 k_p r_0 S T \exp(j2\pi\alpha y_f)\exp\left[\frac{jk}{2f}(x_f^2+y_f^2)\left(2-\frac{d}{f}\right)\right]$$

$$+ \left(\frac{A}{\lambda f}\right)^2 k_p r_0 S^* T \exp(-j2\pi\alpha y_f)\exp\left[\frac{jk}{2f}(x_f^2+y_f^2)\left(-\frac{d}{f}\right)\right]$$

LENS 2 TAKES THE TRANSFORM $U(x_f, y_f)$

$$U_i(X,Y) = \frac{1}{j\lambda f}\, \mathcal{F}\{U(x_f, y_f)\}_{f_x = X/\lambda f,\ f_y = Y/\lambda f}$$

$$= \left(\frac{A}{\lambda f}\right)^3 \frac{k_p}{j}\, \mathcal{F}\left\{T S S^* \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right]\right\}$$

$$+ \frac{r_0^2 A k_p}{j\lambda f}\, \mathcal{F}\left\{T \exp\left[\frac{jk}{2f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right]\right\}$$

$$+ \left(\frac{A}{\lambda f}\right)^2 k_p r_0\, \mathcal{F}\left\{S^* T \exp(-j2\pi\alpha y_f)\exp\left[\frac{-jk}{2f}(x_f^2+y_f^2)\frac{d}{f}\right]\right\}$$

$$- \left(\frac{A}{\lambda f}\right)^2 k_p r_0\, \mathcal{F}\left\{S T \exp(j2\pi\alpha y_f)\exp\left[\frac{jk}{2f}(x_f^2+y_f^2)\left(2-\frac{d}{f}\right)\right]\right\}$$

$$(f_x = X/\lambda f,\ f_y = Y/\lambda f)$$

THE FIRST AND SECOND TERM WILL BE CENTERED ABOUT THE ORIGIN. THE FOURTH AND THIRD TERMS WILL BE SHIFTED ON THE Y AXIS DUE TO THE PHASE TERMS $\exp(\pm j2\pi\alpha Y_f)$

$\Rightarrow$ SHIFT OF $\pm\alpha\lambda f$ (FROM SHIFT THEM. AND SIMILARITY THEM. WITH $f_Y = Y/\lambda f$). IT WILL BE ASSUMED THAT $\alpha\lambda f$ IS LARGE ENOUGH TO SEPARATE THE THIRD AND FOURTH TERMS FROM THE FIRST AND SECOND.

THE TERMS CONTAINING ST AND S^*T ARE OF INTEREST, IN THAT THE MAY BE MANIPULATED TO YIELD CONVOLUTION AND CORRELATION OF t AND s IN THE OUTPUT PLANE BY MEANS OF CHANGING d TO ELIMINATE THE QUADRATIC PHASE TERM. CONSIDER THE FOLLOWING $U(x,Y)$ COMPONENT:

$$\frac{A^2 K_p r_o}{\lambda^2 f^2}\mathcal{F}\left\{S^*\left(\frac{x_f}{\lambda f},\frac{Y_f}{\lambda f}\right)T\left(\frac{x_f}{\lambda f},\frac{Y_f}{\lambda f}\right)\exp(-j\,2\pi\alpha Y_f)\exp\left(\frac{-jk}{2f}(x_f^2+Y_f^2)\frac{d}{f}\right)\right.$$

LETTING $d=0$ YIELDS:

$$\frac{A^2 K_p r_o}{\lambda^2 f^2}\mathcal{F}\left\{S^*\left\{\frac{x_f}{\lambda f},\frac{Y_f}{\lambda f}\right\}T\left(\frac{x_f}{\lambda f},\frac{Y_f}{\lambda f}\right)\exp(-j\,2\pi\alpha Y_f)\right.$$

$$= A^2 K_p r_o\left[s^*(-\lambda f f_x, -\lambda f f_Y - \lambda f\alpha) * t(+\lambda f f_x, +\lambda f f_x - \lambda f\alpha)\right]$$

$$= A^2 K_p r_o\left[s^*(-x, -(Y+\lambda f\alpha)\} * t\{+x, (Y-\lambda f\alpha)\}\right]$$

$$= A^2 K_p r_o\left[s(x, Y-\lambda f\alpha) \star t(x, Y-\lambda f\alpha)\right]$$

THUS, WITH $t(x_o, Y_o)$ PLACED AGAINST LENS 1 YIELDS THE CORRELATION OF t AND s IN THE OUTPUT PLANE SHIFTED $\lambda f\alpha$ IN THE POSITIVE Y DIRECTION (ie FILM'S IMPULSE RESPONSE $= s^*(-x, -(Y+\lambda f\alpha))$, CHARACTERISTIC OF MATCHED FILTER FILTER) (FILTER FILTER?)

CONSIDER THE FOURTH $U(x,Y)$ COMPONENT:

$$-\left(\frac{A}{\lambda f}\right)^2 K_p r_o\,\mathcal{F}\left\{S\,T\exp(j\,2\pi\alpha Y_f)\exp\left[\frac{jk}{2f}(x_f^2+Y_f^2)\left(2-\frac{d}{f}\right)\right]\right.$$

LETTING $d=2f$ YIELDS

$$-\left(\frac{A}{\lambda f}\right)^2 K_p r_o\,\mathcal{F}\left\{S\left(\frac{x_f}{\lambda f},\frac{Y_f}{\lambda f}\right)T\left(\frac{x_f}{\lambda f},\frac{Y_f}{\lambda f}\right)\exp(j\,2\pi\alpha Y_f)\right\}$$

$$= -A^2 K_p r_o\; s(x, Y-\alpha f\lambda) * t(x, Y-\alpha f\lambda)$$

THUS, WITH $t(x_o, Y_o)$ PLACED $2f$ IN FRONT OF LENS 1 YIELDS THE CONVOLUTION OF t AND s. ERGO, THE FILM ACTS AS A FILTER WITH IMPULSE RESPONSE $= s(x, Y-\alpha f\lambda)$

Prob. 7-13 solution:

(a) The reference wave at the recording plane can be given by: $U_r(x,y) = r_0 \exp(-j 2\pi \alpha y)$ where $\alpha = \frac{\sin\theta}{\lambda}$ and θ is the angle of the reference beam.

The field contributed by $s(x,y)$ in the recording plane is given by:

$$U_s(x,y) = \frac{e^{j\frac{k}{2f}(x^2+y^2)}}{\lambda f} \cdot \mathcal{S}\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

where $\mathcal{S}$ is the Fourier transform of $s(x,y)$

The recorded intensity is then:

$$I = |U_s + U_r|^2 = \left[r_0 e^{-jk\sin\theta y} + \frac{e^{j\frac{k}{2f}(x^2+y^2)}}{\lambda f}\mathcal{S}\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\right] \cdot \left[r_0 e^{jk\sin\theta y} + \frac{e^{-j\frac{k}{2f}(x^2+y^2)}}{\lambda f}\mathcal{S}^*\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\right]$$

$$\therefore I = r_0^2 + \frac{1}{(\lambda f)^2}|\mathcal{S}|^2 + \frac{r_0}{\lambda f} e^{-jk\left(y\sin\theta + \frac{x^2}{2f} + \frac{y^2}{2f}\right)}\mathcal{S}^* + \frac{r_0}{\lambda f} e^{jk\left(y\sin\theta + \frac{x^2}{2f} + \frac{y^2}{2f}\right)}\mathcal{S}$$

$= T(x,y)$, the transmittance of the recorded intensity

(b) Assume the input to the second system is $g(x_0, y_0)$. Then the field incident on the filter function $T(x,y)$ is:

$$U_g(x,y) = \frac{1}{\lambda f}\exp\left[j\frac{k}{2f}\left(1-\frac{d}{f}\right)(x^2+y^2)\right] G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$ where G is the Fourier transform of g.

The filter output is $U_g \cdot I = U_g \cdot \tau$

$$= \left[r_0^2 + \frac{1}{(\lambda f)^2} |S'|^2 \right] \frac{1}{\lambda f} \exp\left[i \frac{k}{2f}\left(1 - \frac{d}{f}\right)(x^2+y^2)\right] G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

$$+ \frac{r_0}{(\lambda f)^2} G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) S'^*\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \exp\left[ik \left[-\sin\theta (y) - \frac{x^2}{2f} - \frac{y^2}{2f} + \left(1 - \frac{d}{f}\right)\left(\frac{x^2}{2f} + \frac{y^2}{2f}\right)\right]\right]$$

$$+ \frac{r_0}{(\lambda f)^2} G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) S'\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \exp\left[ik \left[y \sin\theta + \frac{x^2}{2f} + \frac{y^2}{2f} + \left(1 - \frac{d}{f}\right)\left(\frac{x^2}{2f} + \frac{y^2}{2f}\right)\right]\right]$$

The terms in the first line will give on-axis "garbage" terms in the output.

The second line will give the cross-correlation of g & s displaced off-axis proportional to $\sin\theta$ if:

$$-\frac{x^2}{2f} - \frac{y^2}{2f} + \left(1 - \frac{d}{f}\right)\left(\frac{x^2}{2f} + \frac{y^2}{2f}\right) = 0$$

This will hold if $\underline{d = 0}$

The third line will give the convolution of g & s displaced on the other side of the lens axis if:

$$\frac{x^2}{2f} + \frac{y^2}{2f} + \left(1 - \frac{d}{f}\right)\left(\frac{x^2}{2f} + \frac{y^2}{2f}\right) = 0$$

This will hold if $\underline{d = 2f}$

Thus, to get convolution: $d = 2f$
and to get cross-correlation: $d = 0$

The filter output is $U_g \cdot I = U_g \cdot r$

$$= \left[r_0^2 + \frac{1}{(\lambda f)^2}|S|^2\right] \frac{1}{\lambda f} \exp\left[i\frac{k}{2f}\left(1-\frac{d}{f}\right)(x^2+y^2)\right] G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

$$+ \frac{r_0}{(\lambda f)^2} G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) S^*\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \exp\left[ik\left[-\sin\theta(y) - \frac{x^2}{2f} - \frac{y^2}{2f} + \left(1-\frac{d}{f}\right)\left(\frac{x^2}{2f}+\frac{y^2}{2f}\right)\right]\right]$$

$$+ \frac{r_0}{(\lambda f)^2} G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) S\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \exp\left[ik\left[y\sin\theta + \frac{x^2}{2f} + \frac{y^2}{2f} + \left(1-\frac{d}{f}\right)\left(\frac{x^2}{2f}+\frac{y^2}{2f}\right)\right]\right]$$

The terms in the first line will give on-axis "garbage" terms in the output.

The second line will give the cross-correlation of g & s displaced off-axis proportional to $\sin\theta$ if:

$$-\frac{x^2}{2f} - \frac{y^2}{2f} + \left(1-\frac{d}{f}\right)\left(\frac{x^2}{2f}+\frac{y^2}{2f}\right) = 0$$

This will hold if $\underline{\underline{d = 0}}$

The third line will give the convolution of g & s displaced on the other side of the lens axis if:

$$\frac{x^2}{2f} + \frac{y^2}{2f} + \left(1-\frac{d}{f}\right)\left(\frac{x^2}{2f}+\frac{y^2}{2f}\right) = 0$$

This will hold if $\underline{\underline{d = 2f}}$

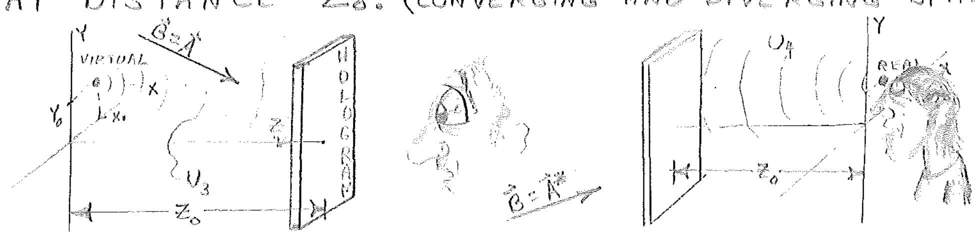
Thus, to get convolution: $d = 2f$

and to get cross-correlation: $d = 0$

8-1) CONSIDER THE FOLLOWING HOLOGRAM OF A POINT SOURCE LOCATED @ (X_0, Y_0) ON THE OBJECT PLANE, PLACED A DISTANCE Z_0 FROM THE FILM. ASSUME PLANAR REFERENCE AND RECONSTRUCTION WAVES, AND THAT THE DEVELOPED PHOTOGRAPHIC PLATE YIELDS A TRANSPARENCY WITH AMPLITUDE TRANSMITTANCE PROPORTIONAL TO EXPOSURE. THE IMPORTANT INFORMATION COMPONENTS OF THE RECONSTRUCTED WAVEFRONT (ANALAGOUS TO EQ (8-24) AND PREVIOUSLY DERIVED IN CLASS) ARE:

$$\vec{U}_3(X,Y) = B'|A|^2 a = B'|A|^2 \vec{a}_0 \exp\left[jk\{Z_0^2 + (X-X_0)^2 + (Y-Y_0)^2\}^{1/2}\right]$$
$$\vec{U}_4(X,Y) = B'|A|^2 a^* = B'|A|^2 \vec{a}_0^{\,*} \exp\left[-jk\{Z_0^2 + (X-X_0)^2 + (Y-Y_0)^2\}\right]$$

WHERE U_3 WAS RECONSTRUCTED WITH A WAVE EQUAL TO THE INITIAL REFERENCE WAVE, AND U_4 ITS CONJUGATE. U_3 REPRESENTS A WEIGHTED VIRTUAL IMAGE OF THE ORIGINAL POINT SOURCE AT A DISTANCE $-Z_0$ FROM THE HOLOGRAM, AND U_4 A REAL IMAGE OF THE PT. SOURCE AT DISTANCE Z_0. (CONVERGING AND DIVERGING SPHERICAL WAVES)



THUS, THE RECONSTRUCTION OF ANY POINT SOURCE WILL LIE A DISTANCE Z_0 FROM THE HOLOGRAM, THE LOCUS OF WHICH FORMS A PLANE PARALLEL WITH THE HOLOGRAM. ANY TRANSPARENCY MAY BE THOUGHT OF AS THE SUPERPOSITION OF POINT SOURCES:

$$t(X_0, Y_0) = \int_{-\infty}^{\infty}\!\!\int t(\xi, \eta)\, \delta(X_0 - \xi, Y_0 - \eta)\, d\xi\, d\eta$$

ERGO, WHEN A HOLOGRAM OF A PLANAR OBJECT IS RECORDED IN A PLANE PARALLEL WITH THE OBJECT, THE RESULTING IMAGES FORM IN PLANES PARALLEL WITH THE HOLOGRAM.

8-2) a) $z_i = \left(\frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \mp \frac{\lambda_2}{\lambda_1 z_o} \right)^{-1}$

$= \left(\frac{1}{\infty} \pm \frac{\lambda_2}{\lambda_1}\frac{1}{\infty} \mp \frac{\lambda_2}{\lambda_1 z_o} \right)^{-1}$

$= \frac{\mp\lambda_1 z_o}{\lambda_2} = \frac{10 \cdot 4880}{6328}$ cm

$= \mp 7.7$ cm (−7.7 cm FOR VIRTUAL, 7.7 cm FOR REAL)

b) $z_i = \left(\frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \mp \frac{\lambda_2}{\lambda_1 z_o} \right)^{-1}$

$= \left(\pm \frac{\lambda_2}{2\lambda_1 z_o} \mp \frac{\lambda_2}{\lambda_1 z_o} \right)^{-1}$

FOR THE VIRTUAL IMAGE

$z_i = \left(\frac{-\lambda_2}{2\lambda_1 z_o} \right)^{-1} = -15.4$ cm (TWICE AS FOR PART a)

FOR THE REAL IMAGE

$z_i = \left(\frac{\lambda_2}{2\lambda_1 z_o} \right)^{-1} = 15.4$ cm (TWICE AS PART a)

NOW $M = \left| 1 - \frac{z_o}{z_r} \mp \frac{\lambda_1 z_o}{\lambda_2 z_p} \right|^{-1}$

$= \left| 1 - \left(\frac{1}{2}\right) \mp 0 \right|^{-1}$

$= 2$ FOR BOTH THE VIRTUAL & REAL IMAGE

(THAT'S IT ?)

8-3) $M = \left| 1 - \frac{z_o}{z_r} \mp \frac{\lambda_1 z_o}{\lambda_2 z_p} \right|^{-1}$

FOR $\lambda_1 = \lambda_2$

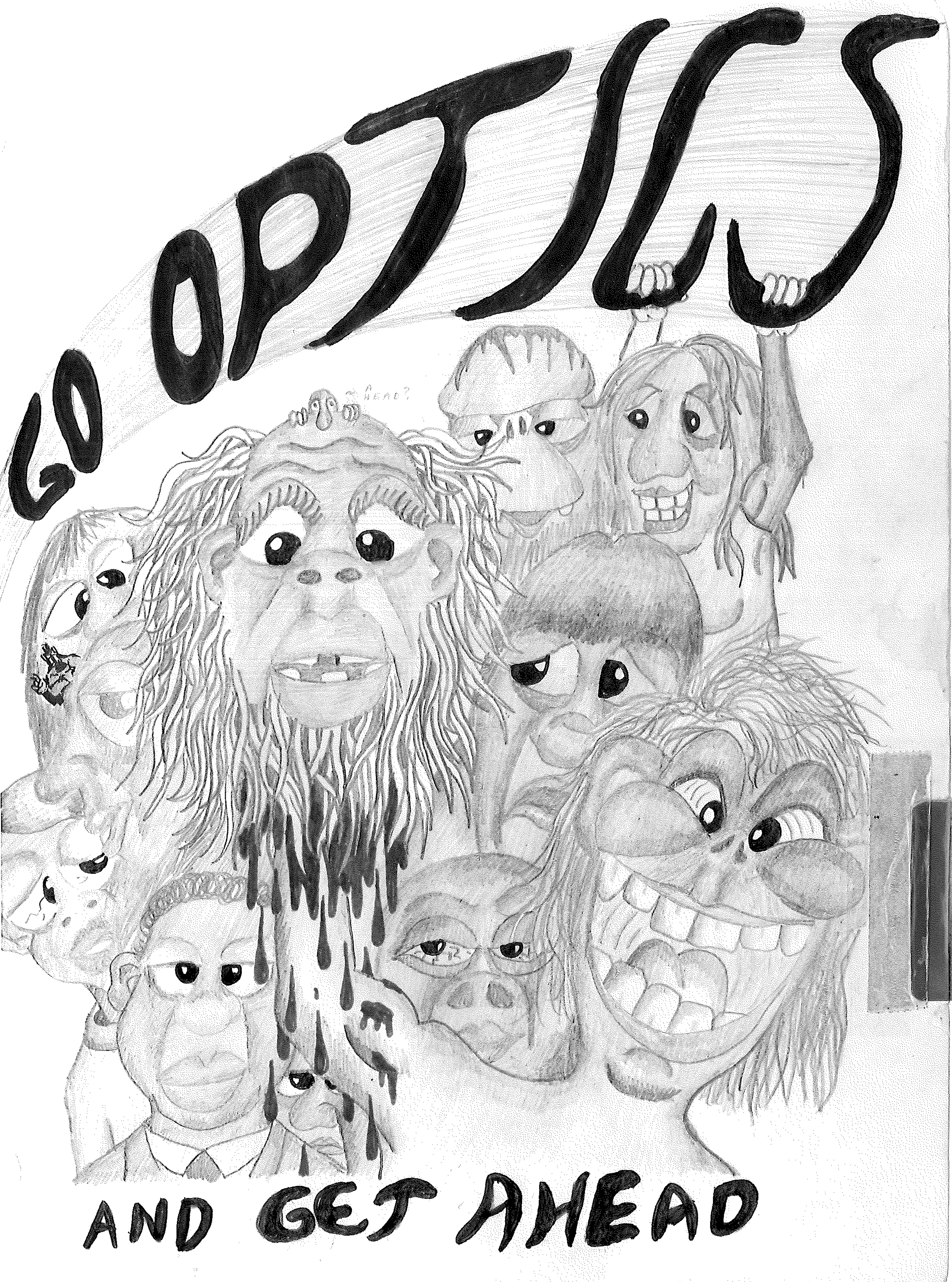
$M = \left| 1 - \frac{z_o}{z_r} \mp \frac{z_o}{z_p} \right|^{-1}$

CASE 1: VIRTUAL IMAGE ; $z_p = z_r$

$M_v = \left| 1 - \frac{z_o}{z_r} + \frac{z_o}{z_p} \right|^{-1} = 1$

CASE 2 ; REAL IMAGE ; $z_p = -z_r$

$M_R = \left| 1 - \frac{z_o}{z_r} - \left(\frac{z_o}{-z_p}\right) \right|^{-1} = 1$


GO OPTICS
A HEAD?
AND GET AHEAD

9-2-75 (T)

EE 5360 - INTRODUCTION TO FOURIER OPTICS AND HOLOGRAPHY

JOHN WALKUP   RM 260B   742-1278   791-0671 (HOME)

TEXT: INTRO. TO FOURIER OPTICS, J.W. GOODMAN, McGRAW HILL, '68

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(FOURIER TRANSFORM)

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GRADING: HOMEWORK 20%, MIDTERM 20%

FINAL 40%, ORAL PRESENTATION 20%

JOURNALS:

OPTICAL SOCIETY OF AMERICA

- J. OPT. SOC. OF AMERICA
- APPLIED OPTICS

OPTICAL ENGINEERING

PHOTOGRAPHIC SCIENCE & ENG.

NOTES:

REVIEW OF 1-D FOURIER TRANSFORM

$\tilde{g}(x) \Leftrightarrow \tilde{G}(f)$: FOURIER TRANSFORM PAIR

$$\begin{cases} \tilde{G}(f) = \int_{-\infty}^{\infty} \tilde{g}(x)\, e^{-j2\pi f x}\, dx \triangleq \mathcal{F}[\tilde{g}(x)] \\ \tilde{g}(x) = \int_{-\infty}^{\infty} \tilde{G}(f)\, e^{j2\pi f x}\, dx \triangleq \mathcal{F}^{-1}[G(f)] \end{cases}$$

IF $x = t$ = TIME, THEN $\tilde{g}(t)$ IS A TEMPORAL FUNCTION, THEN f IS TEMPORAL FREQUENCY (CYCLES/SEC = HZ)

THEOREMS IN 1.D

① LINEARITY: $\mathcal{F}(\cdot)$ IS A LINEAR OPERATOR.

$$\mathcal{F}[\alpha \tilde{g}(x) + \beta \tilde{h}(x)] = \alpha \mathcal{F}[\tilde{g}(x)] + \beta \mathcal{F}[\tilde{h}(x)]$$

$\mathcal{F}^{-1}(\cdot)$ IS ALSO LINEAR.

② SIMILARITY OR SCALING THEM.

IF $\tilde{G}(f) \Leftrightarrow \tilde{g}(x)$

THEN $\mathcal{F}\{\tilde{g}(\alpha x)\} \Longleftrightarrow \frac{1}{|\alpha|} \tilde{G}\left(\frac{f}{\alpha}\right)$

α = SCALAR CONSTANT

③ SHIFT THEOREM

$$\mathcal{F}[\tilde{g}(x-a)] = \tilde{G}(f)\, e^{-j2\pi f a}$$

④ PARCEVAL'S THEM:

$$\int_{-\infty}^{\infty} |\tilde{g}(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{G}(f)|^2 df$$

ie, THE FOURIER TRANSFORM IS AN ENERGY PRESERVING TRANSFORM.

⑤ CONVOLUTION THEOREM

IF $\mathcal{F}[\tilde{g}(x)] = \tilde{G}(f)$ AND $\mathcal{F}[\tilde{h}(x)] = \tilde{H}(f)$,

THEN $\mathcal{F}[\tilde{g}(x) * \tilde{h}(x)] = \mathcal{F}\left[\int_{-\infty}^{\infty} \tilde{g}(\xi)\, \tilde{h}(x-\xi)\, d\xi\right]$

$$= \tilde{G}(f)\, \tilde{H}(f)$$

OR $\tilde{g}(x) * \tilde{h}(x) = \mathcal{F}^{-1}[\tilde{G}(f)\, \tilde{H}(f)]$

$\tilde{H}(f) \Rightarrow$ TRANSFER FUNCTION

⑥ AUTOCORRELATION THEM.

IF $\mathcal{F}[\tilde{g}(x)] = \tilde{G}(f)$

THEN $\mathcal{F}[\tilde{g} \star \tilde{g}] = \mathcal{F}\left[\int_{-\infty}^{\infty} \tilde{g}(\xi)\, \tilde{g}^*(\xi - x)\, d\xi\right]$

$$= |G(f)|^2$$

SIMILARLY: $\mathcal{F}\{|g(x)|^2\} = \mathcal{F}\left[\int_{-\infty}^{\infty} \tilde{G}(\xi)\, \tilde{G}^*(\xi - f)\, d\xi\right]$

$$= \tilde{G} \star \tilde{G}$$

⑦ FOURIER INTEGRAL THEOREM:

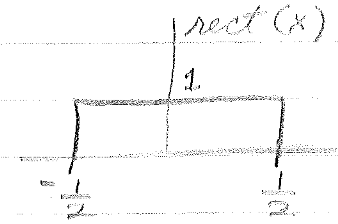
AT EACH POINT OF CONTINUITY OF $\tilde{g}(x)$, THEN $\mathcal{F}^{-1}\mathcal{F}[\tilde{g}(x)] = \tilde{g}(x)$, WHILE AT EACH PT WHEN $\tilde{g}(x)$ IS DISCONTINUOUS,

$$\mathcal{F}^{-1}[\mathcal{F}\{g(x)\}] = \frac{1}{2}[\tilde{g}(x+) + \tilde{g}(x-)]$$

ELEMENTARY FUNCTIONS

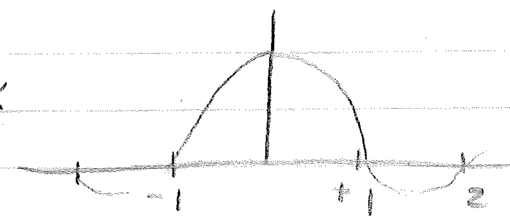
(a) THE RECTANGLE FUNCTION

$$\mathrm{rect}(x) \triangleq \begin{cases} 1 & ; |x| < \frac{1}{2} \\ \frac{1}{2} & ; |x| = \frac{1}{2} \\ 0 & ; |x| > \frac{1}{2} \end{cases}$$



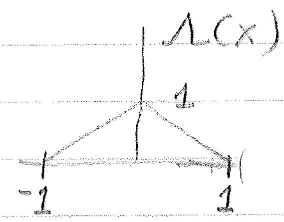
(b) THE SINC FUNCTION

$$\mathrm{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}$$



(c) THE TRIANGLE FUNCTION

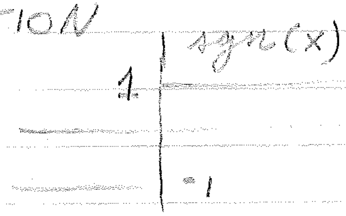
$$\Lambda(x) \triangleq \begin{cases} 1 - |x| & ; |x| \le 1 \\ 0 & ; |x| > 1 \end{cases}$$



$$\Lambda(x) = \mathrm{rect}(x) * \mathrm{rect}(x)$$

(d) THE SGN ("SIGUN") FUNCTION

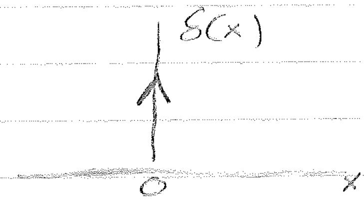
$$\mathrm{sgn}(x) \triangleq \begin{cases} 1 & ; x > 0 \\ 0 & ; x = 0 \\ -1 & ; x < 0 \end{cases}$$



(e) THE DIRAC DELTA, $\delta(x)$

$$\delta(x) = \begin{cases} \infty & ; x = 0 \\ 0 & ; x \neq 0 \end{cases}$$

$$\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1 \quad \forall\, \epsilon > 0$$



PROPERTIES:

(1) SIFTING PROPERTY $\int_{-\infty}^{\infty} \tilde{v}(x)\,\delta(x-a)\,dx = \tilde{v}(a)$

IF $\tilde{v}(x)$ IS CONTINUOUS @ $x = a$

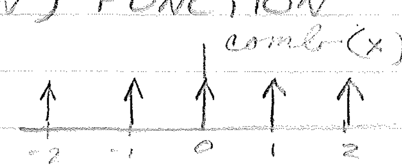
(2) $\delta(ax) = \frac{1}{|a|}\delta(x)$

(3) $\delta(x) = \delta(-x)$

(4) $\tilde{v}(x)\,\delta(x) = \tilde{v}(0)\,\delta(x)$ IF $\tilde{v}(x)$ IS CONTINUOUS AT $x = 0$

(f). THE "COMB" (SAMPLING TRAIN) FUNCTION

$$\mathrm{comb}(x) \triangleq \sum_{n=-\infty}^{\infty} \delta(x-n)$$



9-4-75 (THURS)

USEFUL 1-D FOURIER TRANSFORM PAIRS

$$\tilde{g}(x) \Longleftrightarrow G(f)$$

1. $e^{-\pi x^2} \Longleftrightarrow e^{-\pi f^2}$
2. $\mathrm{rect}(x) \Longleftrightarrow \mathrm{sinc}(f)$
3. $\Lambda(x) \Longleftrightarrow \mathrm{sinc}^2(f)$
4. $\mathrm{sinc}(x) \Longleftrightarrow \mathrm{rect}(f)$
5. $\mathrm{sinc}^2(x) \Longleftrightarrow \Lambda(f)$
6. $\delta(x) \Longleftrightarrow 1$
7. $1 \Longleftrightarrow \delta(f)$
8. $e^{-j\pi x} \Longleftrightarrow \delta(f-\frac{1}{2})$
9. $\mathrm{sgn}(x) \Longleftrightarrow \frac{1}{j\pi f}$
10. $\frac{1}{j\pi f} \Longleftrightarrow \mathrm{sgn}(f)$
11. $\mathrm{comb}(x) \Longleftrightarrow \mathrm{comb}(x)$
12. $\cos \pi x \Longleftrightarrow \frac{1}{2}\delta(f-\frac{1}{2}) + \frac{1}{2}\delta(f+\frac{1}{2})$
13. $\sin \pi x \Longleftrightarrow \frac{1}{j2}\delta(f-\frac{1}{2}) - \frac{1}{j2}\delta(f+\frac{1}{2})$
14. $e^{-|x|} \Longleftrightarrow \frac{2}{1+(2\pi f)^2}$

TWO DIMENSIONAL TRANSFORM

$$\tilde{g}(x, y) \Leftrightarrow \tilde{G}(f_x, f_y)$$

$$\tilde{G}(f_x, f_y) = \int_{-\infty}^{\infty}\int \tilde{g}(x, y) e^{-j2\pi[f_x x + f_y y]} dx\,dy$$
$$= \mathcal{F}[\tilde{g}(x, y)]$$
$$\tilde{g}(x, y) = \int_{-\infty}^{\infty}\int \tilde{G}(f_x, f_y) e^{j2\pi[f_x x + f_y y]} df_x\,df_y$$
$$= \mathcal{F}^{-1}[\tilde{G}(f_x, f_y)]$$

TEMPORAL FREQUENCY: $g(t) \Leftrightarrow G(f) \Rightarrow [f] = \frac{\text{CYCLE}}{\text{SEC}}$

SPATIAL FREQUENCY: $f = \frac{\text{LINES}}{\text{MM}}$

TWO DIMENSION FOURIER THEOREMS

① LINEARITY THEOREM

$$\mathcal{F}[\alpha f(x,y) + \beta h(x,y)] = \alpha \mathcal{F}[f(x,y)] + \beta \mathcal{F}[h(x,y)]$$

② SIMILARITY (OR SCALING) THEOREM

$$\mathcal{F}[g(ax, \beta y)] = \frac{1}{|\alpha\beta|} \hat{G}\left(\frac{f_x}{\alpha}, \frac{f_y}{\beta}\right)$$

③ SPATIAL SHIFT THEOREM: IF $g(x,y) \Leftrightarrow G(f_x, f_y)$

THEN $\mathcal{F}[g(x-a, y-b)] = G(f_x, f_y) e^{-j2\pi(f_x a + f_y b)}$

④ PARCEVAL'S THEOREM: $g(x,y) \Leftrightarrow G(f_x, f_y)$

$$\int_{-\infty}^{\infty}\int |\hat{g}(x,y)|^2 dx\,dy = \int_{-\infty}^{\infty}\int |\tilde{G}(f_x, f_y)|^2 df_x\,df_y$$

⑤ CONVOLUTION THEOREM: $H(f_x, f_y) \Leftrightarrow h(x, y)$

$$\mathcal{F}[\hat{g}(x,y) * h(x,y)] = G(f_x, f_y) H(f_x, f_y)$$

$$g(x,y) * h(x,y) = \int_{-\infty}^{\infty} g(\xi, \eta) h(x-\xi, y-\eta) d\xi\,d\eta$$

⑥ AUTOCORRELATION THEOREM

$$\mathcal{F}[g \star g] = |\tilde{G}(f_x, f_y)|^2$$

$$g \star g = \int_{-\infty}^{\infty}\int g(\xi, \eta) g^*(\xi - x, \eta - y) d\xi\,d\eta$$

BY SYMMETRY: $\mathcal{F}\{|\hat{g}(x,y)|^2\} = G(f_x, f_y) \star G(f_x, f_y)$

⑦ FOURIER INTEGRAL THEOREM: AT EACH POINT OF CONTINUITY OF $\hat{g}(x,y)$, $\mathcal{F}^{-1}\mathcal{F}[g(x,y)] = g(x,y)$ WHILE AT EACH POINT WHERE g IS DISCONT. $\mathcal{F}^{-1}\mathcal{F}[\cdot]$ YIELDS THE SPATIAL ARGUMENT IN g ABOUT THE POINT.

TWO-DIMENSIONAL SEPERABLE FUNCTIONS

$G(f_X, f_Y) = \int\int_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi[f_X x + f_Y y]}\, dy$

IF $g(x,y) = g_X(x)\, g_Y(y)$, THEN g IS SEPERABLE IN CARTESIAN COORDINATES, THEN

$G(f_X, f_Y) = \int_{-\infty}^{\infty} g_X(x)\, e^{-j2\pi f_X x}\, dx \int_{-\infty}^{\infty} g_Y(y)\, e^{-j2\pi f_Y y}\, dy$

$= G_X(f_X)\, G_Y(f_Y)$

SO THE 2-D FOURIER TRANSFORMS IS NOW THE PRODUCT OF TWO 1-D TRANSFORMS.

CONSIDER POLAR CO-ORDINATES $r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$g(r,\theta)$ IS SEPERABLE IN POLAR COOR. IF

$g(r,\theta) = g_R(r)\, g_\Theta(\theta)$

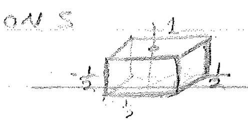
IF g HAS CIRCULAR SYMMETRY, THEN

$g(r,\theta) = g_R(r)$ ie $g_\Theta(\theta) = 1$

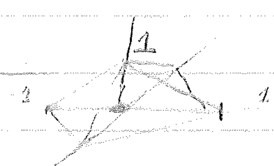
AND TAKING THE TWO-D FOURIER TRANSFORM IS EQUIVAVENT TO A SINGLE INTEGRAL.

SOME USEFUL ELEMENTARY FUNCTIONS

(a) $\text{rect}(x)\, \text{rect}(y) = \text{rect}(x,y)$



(b) $\Lambda(x)\, \Lambda(y) = \Lambda(x,y)$



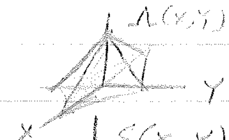
9·9·75 (TUES)

H.W. GOODMAN, CH 2. (3a, 3c, 4, 5, 6, 7, 8)

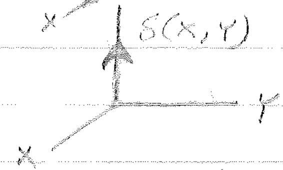
USEFUL 2-d ELEMENTARY FUNCTIONS

① $\mathrm{rect}(x)\,\mathrm{rect}(Y) = \mathrm{rect}(x, Y)$

② $\Lambda(x)\,\Lambda(Y) = \Lambda(x, Y)$

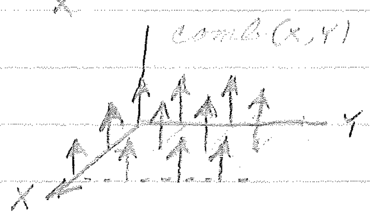


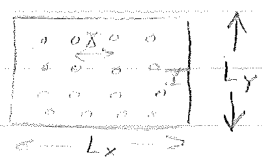
③ $\delta(X)\,\delta(Y) = \delta(X, Y)$



④ $\mathrm{comb}(X)\,\mathrm{comb}(Y)$

$$= \mathrm{comb}(X, Y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \delta(x-n, Y-m)$$

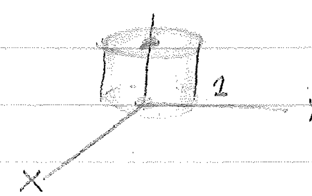




$$= \mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{Y}{Y}\right)\mathrm{rect}\left(\frac{X}{L_X}, \frac{Y}{L_Y}\right)$$

⑤ THE CIRCLE FUNCTION

$$\mathrm{circ}(\sqrt{X^2+Y^2}) = \begin{cases} 1, & \sqrt{X^2+Y^2} < 1 \\ 0, & \text{OTHERWISE} \end{cases}$$



⑥ DIRAC DELTA PROPERTIES

a. $\delta(x, Y) = \begin{cases} \infty & ; X = Y = 0 \\ 0 & ; \text{OTHERWISE} \end{cases}$

b. $\int_{-\epsilon}^{\epsilon}\int \delta(x,Y)\,dx\,dY = 1 \;\forall\; \epsilon > 0$

c. $\int_{-\infty}^{\infty}\int g(\xi, \eta)\,\delta(x-\xi, Y-\eta)\,d\xi\,d\eta = g(x, Y)$

d. $\delta(ax, bY) = \frac{1}{|ab|}\,\delta(x, Y)$

e. $\delta(x, Y) = \lim_{N\to\infty} N^2\,\mathrm{rect}(NX, NY)$

$= \lim N^2\, e^{-N^2\pi(X^2+Y^2)}$

USEFUL TWO DIMENSIONAL FOURIER TRANSFORM PAIRS

$g(x,y) \Leftrightarrow G(f_x, f_y)$

1. $e^{-\pi(x^2+y^2)} \Leftrightarrow e^{-\pi(f_x^2+f_y^2)}$
2. $\mathrm{rect}(x,y) \Leftrightarrow \mathrm{sinc}(f_x, f_y)$
3. $\Lambda(x)\Lambda(y) \Leftrightarrow \mathrm{sinc}^2(f_x, f_y)$
4. $\delta(x,y) \Leftrightarrow 1$
5. $e^{j\pi(x+y)} \Leftrightarrow \delta(f_x - \frac{1}{2}, f_y - \frac{1}{2})$
6. $1 \Leftrightarrow \delta(f_x, f_y)$
7. $\mathrm{sgn}(x,y) \Leftrightarrow (j\pi f_x)^{-1}(j\pi f_y)^{-1}$
8. $\mathrm{comb}(x,y) \Leftrightarrow \mathrm{comb}(f_x, f_y)$

CIRCULARLY SYMMETRIC FUNCTIONS

$g(x,y) = g_x(x)\, g_y(y) \Leftarrow$ SEPERABLE IN CARTESIAN

$g(r,\theta) = g_R(r)\, g_\Theta(\theta) \Leftarrow$ SEPERABLE IN POLAR COOR

$= g_R(r) \Leftarrow$ CIRCULARLY SYMMETRIC

$$G(f_x,f_y) = \int\!\!\int_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\, dy$$

NOW $r = \sqrt{x^2+y^2}$ OR $x = r\cos\theta$

$\theta = \tan^{-1}\theta$ $\qquad y = r\sin\theta$

IN f_x, f_y PLANE, LET

$\rho = \sqrt{f_x^2 + f_y^2}$ OR $f_x = \rho\cos\phi$

$\phi = \tan^{-1}(f_y/f_x)$ $\qquad f_y = \rho\sin\phi$

LET $\tilde{G}(\rho,\phi) \triangleq \mathcal{F}[g]$

MAKING SUBSTITUTIONS GIVES

$$G(\rho,\phi) = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r\, \tilde{g}_R(r)\, e^{-j2\pi r\rho[\cos\theta\cos\phi + \sin\theta\sin\phi]}$$

$$= \int_0^{\infty} dr\, r\, g_R(r) \int_0^{2\pi} d\theta\, e^{-j2\pi r\rho\cos(\theta-\phi)}$$

USE IDENTITY: $J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ja\cos(\theta-\phi)}\, d\theta$

$\Rightarrow J_0 =$ ZERO ORDER BESSEL FUNCTION OF FIRST KIND

THUS $G(\rho,\phi) = G(\rho) = 2\pi \int_0^{\infty} r\, \tilde{g}_R(r)\, J_0(2\pi r\rho)\, dr$

= THE FOURIER-BESSEL TRANSFORM

OR HANKEL TRANSFORM OF $\tilde{g}_R(x)$

$\Rightarrow G(\rho) = \mathcal{B}[g_R(r)]$

$g_R(r) = \mathcal{B}^{-1}[G(\rho)] = 2\pi \int_0^{\infty} \rho\, G(\rho)\, J_0(2\pi r\rho)\, d\rho$

FOURIER-BESSEL TRANSFORM THEOREMS

$\mathcal{B}[g_R(\alpha r)] = \frac{1}{\alpha^2} G_0(\rho/\alpha)$

EXAMPLE: EVALUATE $\mathcal{B}[circ(r/a_0)]$

$\mathcal{B}[circ(r)] = 2\pi \int_0^1 r J_0(2\pi r\rho)\,dr$

CHANGE OF VARIABLES: $r' = 2\pi r\rho$

$\Rightarrow dr' = 2\pi\rho\,dr$. USE IDENTITY THAT

$\int_0^x \xi J_0(\xi)\,d\xi = J_1(x)$

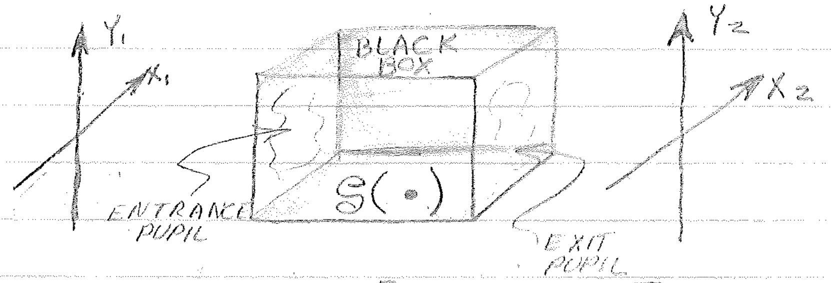
J_1 = BESSEL FUNCTION OF 1ST KIND ORDER 1

GIVES $\mathcal{B}[circ(r)] = \frac{1}{2\pi\rho^2}\int_0^{2\pi\rho} r' J_0(r')\,dr'$

$= \frac{1}{\rho} J_1(2\pi\rho)$ (= "SOMBRARO" FUNCTION)

$\therefore \mathcal{B}[circ(r/a_0)] = \frac{a_0}{\rho} J_1(2\pi\rho a_0)$

LINEAR SYSTEM'S THEORY



$g_2(x_2, y_2) = \mathcal{S}[g_1(x_1, y_1)]$

$g_1(x_1, y_1) = \int_{-\infty}^{\infty}\int g_1(\xi,\eta)\,\delta(x_1-\xi, y_1-\eta)\,d\xi\,d\eta$

LET $h_2(x_2, y_2; x_1, y_1) = \mathcal{S}[\delta(x_1 - x_2, y_1 - y_2)]$

= IMPULSE RESPONSE = POINT-SPREAD FUNCTION

$h_2(x_2, y_2; \xi, \eta) = \mathcal{S}[\delta(x - \xi, y - \eta)]$

$\mathcal{S}[g_1(x_1, y_1)] = \mathcal{S}[\int_{-\infty}^{\infty}\int g_1(\xi,\eta)\,\delta(x_1-\xi, y_1-\eta)\,d\xi\,d\eta]$

$= g_2(x_2 y_2) = \int_{-\infty}^{\infty}\int g(\xi,\eta)\,\mathcal{S}[\delta(x_1-\xi, y_1-\eta)]\,d\xi\,d\eta$

$= \int_{-\infty}^{\infty}\int g(\xi,\eta)\,h(x_2, y_2; \xi, \eta)\,d\xi\,d\eta$

= TWO DIMENSIONAL SUPERPOSITION INTEGRAL

IF $h(x_2, y_2; \xi, \eta) = h(x_2 - \xi, y - \eta)$

THEN WE GOT DA CONVOLUTION INTEGRAL:

$g_2(x_2, y_2) = \int_{-\infty}^{\infty}\int g(\xi,\eta)\,h(x_2 - \xi, y_2 - \eta)\,d\xi\,d\eta$

AND THE SYSTEM IS SPACE-INVARIANT OR ISOPLANATIC PATCHES.

$g_2(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2)$

$H(f_X, f_Y) = \mathcal{F}[h(x_2, y_2)] \Rightarrow$ TRANSFER FUNCTION

10-11-75 (THURS.)

REVIEW: INVARIANT LINEAR SYSTEMS

$\tilde{g}_2(x_2, y_2) = \int_{-\infty}^{\infty}\int g_1(\xi, \eta)\, h(x_2, y_2; \xi, \eta)\, d\xi\, d\eta$

SUPERPOSITION FOR LINEAR SYSTEM

FOR $h(x_2, y_2; \xi, \eta) = h(x_2 - \xi, y_2 - \eta)$ THE SYSTEM IS INVARIANT AND THE SUPERPOSITION INTEGRAL BECOMES CONVOLUTION

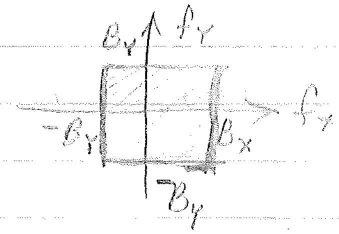
$g_2(x_2, y_2) = \int_{-\infty}^{\infty}\int g_1(\xi, \eta)\, h(x_2 - \xi, y_2 - \eta)\, d\xi\, d\eta$

RECALL THAT FOR CIRCUITS: $h(t) = 0$ FOR $t < 0$. i.e. CAUSALITY. THERE IS NO CAUSALITY PROBLEMS IN OPTICS.

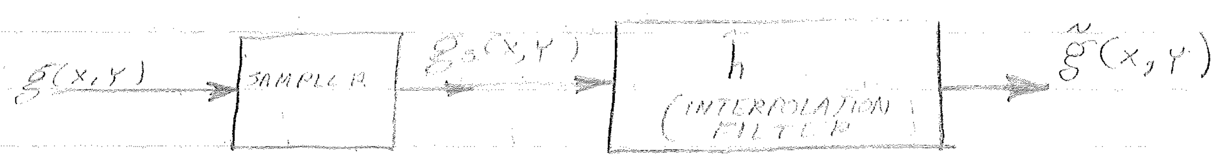
SAMPLING THEOREM IN TWO DIMENSIONS

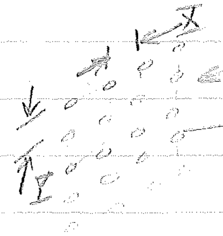
(WHITTIKAR-SHANNON SAMPLING THEOREM)

WE NEED A BAND LIMITED FUNCTION



$G(f_X, f_Y) = 0$ FOR $|f_X| > B_X,\ |f_Y| > B_Y$



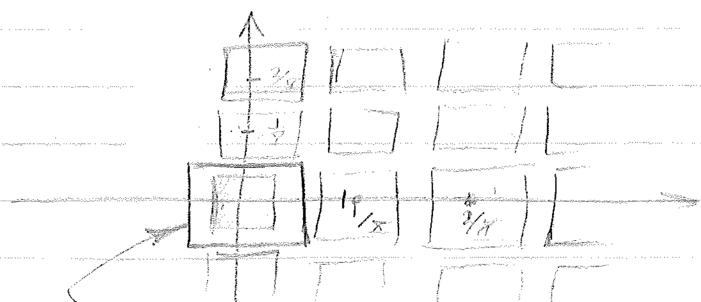


SAMPLING GRID $\longrightarrow \mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)$

SO SAMPLED FUNCTION IS

$$g_s(x,y) = g(x,y)\,\mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)$$

$$\mathcal{F}[g_s(x,y)] = G_s(f_x, f_y)$$
$$= \mathcal{F}[g(x,y)] * \left(\mathrm{comb}\,\frac{x}{X}\,\mathrm{comb}\,\frac{y}{Y}\right)$$
$$= G(f_x, f_y) * XY\,\mathrm{comb}\,Xf_x\,\mathrm{comb}\,Yf_y$$
$$= G(f_x, f_y)\sum_m \sum_n \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$
$$= \sum_n \sum_m G\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$



LOW PASS FILTER $H(f_x, f_y) = \mathrm{rect}\left[\frac{f_x}{2B_x}, \frac{f_y}{2B_y}\right]$

TO AVOID OVERLAPPING (ALIASING)

$\frac{1}{X} \geq 2B_x \qquad \frac{1}{Y} \geq 2B_y$ ← NYQUIST SAMPLING RATES

$X \leq \frac{1}{2B_x} \qquad Y \leq \frac{1}{2B_y}$

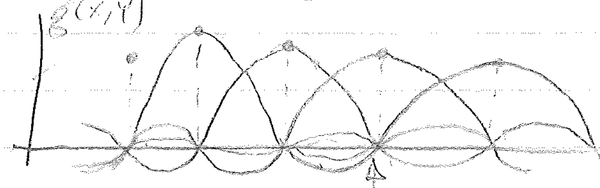
$$H(f_x, f_y) = \mathrm{rect}\left[\frac{f_x}{2B_x}, \frac{f_y}{2B_y}\right]$$
$$\Rightarrow h(x,y) = \mathcal{F}^{-1}[H(f_x, f_y)]$$
$$= 4B_xB_y\,\mathrm{sinc}\,(2B_x x, 2B_y y)$$

THEN: $g(x,y) = g_s(x,y) * h(x,y)$

$$= h(x,y) * \left[g(x,y)\,\mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)\right]$$

LET $X = \frac{1}{2B_x}$ AND $Y = \frac{1}{2B_y}$

$$\Rightarrow g(x,y) = \sum_n \sum_m g\left(\frac{n}{2B_x}, \frac{m}{2B_y}\right)\mathrm{sinc}\left[2B_x\left(x - \frac{n}{2B_x}\right), 2B_y\left(y - \frac{m}{2B_y}\right)\right]$$



NOTE
ZERO CROSSOVER

AN ADDITIONAL HOMEWORK PROBLEM (DUE 9-16-75 TUES

WE HAVE DEFINED THE POINT-SPREAD FUNCTION, $h(x,y)$, OF AN OPTICAL SYSTEM AS A RESPONSE TO $\delta(x,y)$. WE MAY ALSO DEFINE THE LINE SPREAD FUNCTION, $\tilde{\delta}(x)$ AS THE RESPONSE TO A LINE SOURCE.

a. SHOW THAT $\tilde{\delta}(x) = \int_{-\infty}^{\infty} h(x,y)\,dy$

b. FIND THE LINE SPREAD FUNCTIONS, $\tilde{\delta}(x)$ FOR SYSTEMS WITH IMPULSE RESPONSES:

$$h(x,y) = \mathrm{rect}(x)\,\mathrm{rect}(y)$$

$$h(x,y) = \mathrm{circ}(r) \quad \text{WHERE } r = \sqrt{x^2+y^2}$$

c. SHOW THAT THE POINT SPREAD FUNCTION OF A SYSTEM CAN BE DETERMINED IF THE LINE-SPREAD FUNCTIONS ARE KNOWN FOR LINE SOURCES WITH ALL POSSIBLE ORIENTATIONS THROUGH THE ORIGIN OF THE (x,y) PLANE

<u>HINT</u>: (USE FREQUENCY DOMAIN REASONING)

9-16-75 (TUES)

H.W. # 2: GOODMAN CH 2 (9, 11), CH 3 (1, 2, 3, 4)

SCALAR DIFFRACTION THEORY

$$u(p,t) = u(P)\cos[2\pi\nu t + \phi(P)]$$

$$\nu \sim 10^{15}\ \mathrm{HZ}$$

$$\tilde{U}(p,t) = \underbrace{u(p)\, e^{-j\phi(p)}}_{\text{SPATIAL VARIATION}}\ \underbrace{e^{-j2\pi\nu t}}_{\text{TEMPORAL VARIATION}}$$

TWO BASIC ASSUMPTIONS

(1) SPACIAL VARIATIONS IN THE DIFFRACTING STRUCTURE (e.g. APERTURE) ARE COARSE COMPARED WITH λ.

$$\lambda = \frac{c}{\nu}$$

(2) THE DIFFRACTED FIELD MUST NOT BE OBSERVED TO CLOSE TO THE DIFFRACTING STRUCTURE.

HELMHOLTZ EQUATION

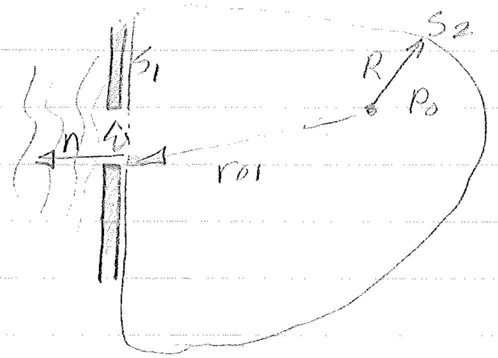
$(\nabla^2 + k)\,\tilde{U} = 0 \doteq$ TIME INDEPENDENT EQUATION IN FREE SPACE

GREEN'S THEOREM

$$\iiint_V [G\nabla^2 U - U\nabla^2 G]\,dv = \iint_S \left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}\right)ds$$

G IS A GREENS FUNCTION

U IS COMPLEX LIGHT FIELD.



$$U(P_0) = \frac{1}{4\pi}\iint_S \left\{\frac{\partial U}{\partial n}\left[\frac{e^{jkr_{01}}}{r_{01}}\right] - U\frac{\partial}{\partial n}\frac{e^{jkr_{01}}}{r_{01}}\right\}ds$$

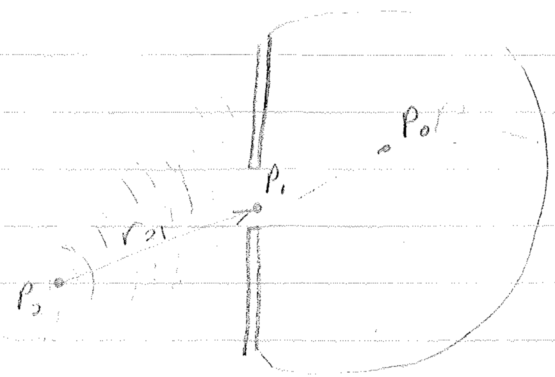
KIRCHOFF BOUNDRY CONDITIONS

① ACCROSS Σ, U AND $\frac{\delta U}{\delta n}$ ARE SAME AS IF NO SCREEN PRESENT.

② ACCROSS S_1, U AND $\frac{\delta U}{\delta n} = 0$

THEN

$$U(P_0) = \frac{1}{4\pi} \iint_{\Sigma} \left[\frac{\delta U}{\delta n} \tilde{G} - U \frac{\delta \tilde{G}}{\delta n} \right] ds$$



WILL GET @ P_1, $U(P_1) = \frac{A e^{jkr_{21}}}{r_{21}}$

ASSUME THE SOMMERFIELD RADIATION COND.

$$\lim_{R \to \infty} R\left(\frac{\delta U}{\delta n} - jkU\right) = 0$$

GIVES FORMULA:

$$U(P_0) = \frac{A}{j\lambda} \iint_{\Sigma} \frac{e^{jk(r_{21}+r_{01})}}{r_{21} r_{01}} \left[\frac{\cos(\bar{n}, \bar{r}_{01}) - \cos(\bar{n}, \bar{r}_{21})}{2} \right] ds$$

THIS IS FRESNEL KIRCHHOFF DIFFRACTION FORMULA.

$$U(P_0) = \iint_{\Sigma} \tilde{U}'(P_1) \frac{e^{jkr_{01}}}{r_{01}} \, ds$$

OBLIQUITY FACTOR

$$\Rightarrow U'(P_1) = \frac{1}{j\lambda} \frac{A e^{jr_{21}k}}{r_{21}} \left[\frac{\cos(\ ) - \cos(\ )}{2} \right] ds$$

SIMILAR EXPRESSION FOR RALEIGH - SOMMERFIELD

$$U(P_0) = \frac{A}{j\lambda} \iint_\Sigma \frac{e^{jk(r_{21}+r_{01})}}{r_{21} r_{01}} \cos(\vec{n}, \vec{r}_{01})$$

BOTH HAVE THE FORM:

$$\hat{U}(P_0) = \iint_\Sigma U_1(P_1)\, \hat{h}(p_0, p_1)\, ds$$

$$h(P_0, P_1) = \frac{1}{j\lambda} \frac{e^{jkr_{01}}}{r_{01}} \cos(\vec{n}, \vec{r}_{01})$$

$$U(P_1) = A e^{jkr_{21}} / r_{21}$$

THIS IS THE HUYGEN'S FRESNEL PRINCIPLE.

9-18-75 (THURS.)

U(X,Y,0) U(X,YZ)

$$U(x, Y) = \int_{-\infty}^{\infty}\!\!\int \hat{A}_0(f_x, f_Y)\, e^{j2\pi(f_x x + f_Y Y)}\, df_x\, df_Y$$

WHERE $A_0(f_x, f_Y) = \mathcal{F}[U(x, Y, 0)]$

$$A_0(f_x, f_Y) = \int_{-\infty}^{\infty}\!\!\int U(x, Y, 0)\, e^{-j2\pi(f_x X + f_Y Y)}\, dx\, dY$$

FOR A PLANE WAVE PROPAGATING WITH DIRECTION CONSINE $\alpha, \beta, \gamma \ni \alpha^2 + \beta^2 + \gamma^2 = 1$

THEN

$$B(x, Y, Z) = e^{j\frac{2\pi}{\lambda}(\alpha X + \beta Y + \gamma Z)}$$

AT $Z = 0$, $B(x, Y, 0) = e^{j\frac{2\pi}{\lambda}(\alpha X + \beta Y)}$

COMPARE WIT $U(X, Y, 0)$

$$\Rightarrow f_x = \frac{\alpha}{\lambda},\quad f_Y = \frac{\beta}{\lambda}$$

THEN $A(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda})$ = ANGULAR SPECTRUM OF $U(X, Y, 0)$

WHAT ABOUT $A(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, z)$?

TO CALCULATE $U(X,Y,Z)$ WE GOTTA SOLVE HELMHOLTZ'S EQ: $\nabla^2 U + k^2 U = 0$

SOLUTION (FOR $A(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, Z)$) IS

$$A(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda}, Z) = A_0(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda})\, e^{j\frac{2\pi}{\lambda}\gamma Z},\quad \gamma^2 = 1 - \alpha^2 - \beta^2$$

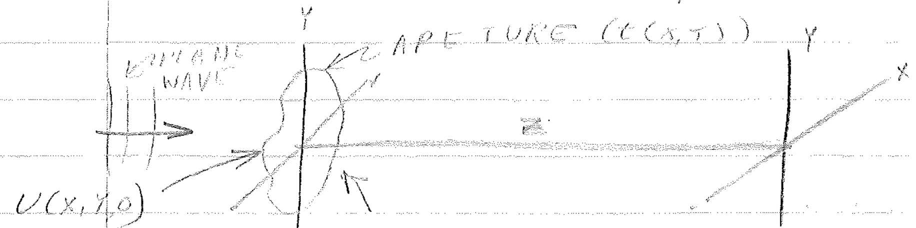
$$\frac{A(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, Z)}{A_0(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda})} = e^{j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,Z} \Leftarrow \text{TRANSFER FUNCTION}$$

FOR $\alpha^2 + \beta^2 < 1$, $e^{j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,Z}$ IS A PROP. WAVE

FOR $\alpha^2 + \beta^2 > 1$, THE WAVE IS ATTENUATED EXPONENTIALLY (EVANESCENT WAVES)

$$\alpha^2 + \beta^2 < 1 \Rightarrow f_x^2 + f_y^2 < \frac{1}{\lambda^2} \Rightarrow \sqrt{f_x^2 + f_y^2} < \frac{1}{\lambda}$$

$$\alpha^2 + \beta^2 > 1 \Rightarrow \sqrt{f_x^2 + f_y^2} > \frac{1}{\lambda}$$

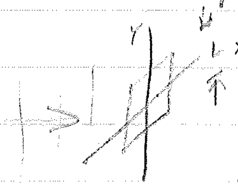


$$U_t(X,Y,0) = U_i(X,Y,0)\,\hat{t}(X,Y)$$

$$A(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda}) = \delta(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda})$$

$$A_t(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda}) = \delta(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda}) * T(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda}) = T(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda})$$

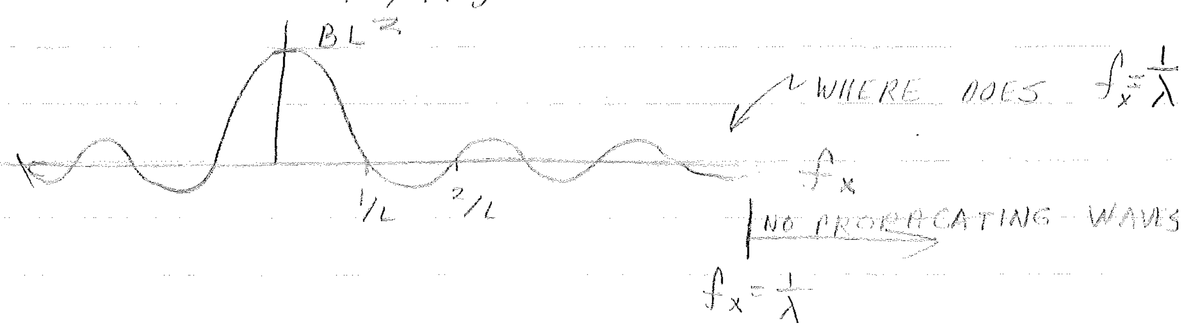
EXAMPLE:



LET $U_t(X,Y,0) = B\,\mathrm{rect}(\frac{X}{L}, \frac{Y}{L})$

$$A_0(f_x, f_y) = BL^2\,\mathrm{sinc}\,Lf_x\,\mathrm{sinc}\,Lf_y$$

$$\Rightarrow A_0(\tfrac{\alpha}{\lambda}, \tfrac{\beta}{\lambda}) = BL^2\,\mathrm{sinc}\,\frac{L\alpha}{\lambda}\,\mathrm{sinc}\,\frac{L\beta}{\lambda}$$



EVANESCENT CUT-OFF:

$f_x^2 + f_y^2 \quad \frac{1}{\lambda^2}$ OR $\alpha^2 + \beta^2 = 1$

FOR $L \gg \lambda$, MORE SPATIAL FREQUENCIES GET THROUGH.

FOR $L \sim \lambda$, CUTOFF AROUND $\frac{1}{L}$.

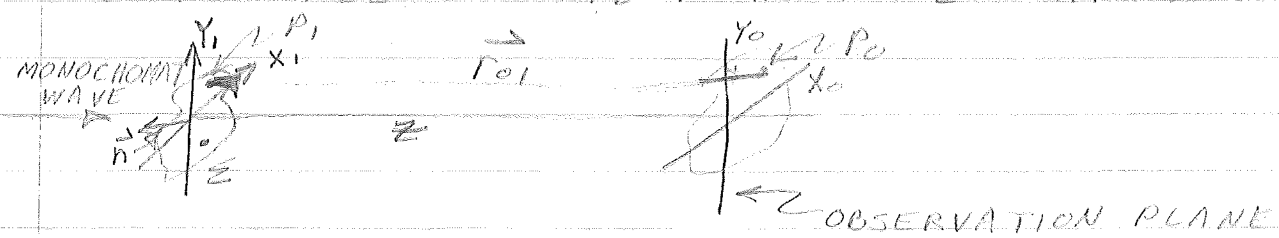
SPECTRAL COMPONENTS AT SPATIAL FREQUENCIES $> \frac{1}{\lambda}$ WON'T PROPAGATE. IF $\frac{1}{L} \ll 1/\lambda$, THEN THE APERTURE WILL PASS A WIDE RANGE OF SPATICAL FREQS. HOWEVER, IF $\frac{1}{L} > \frac{1}{\lambda}$ (ie $L < \lambda$), THEN $f_X^2 + f_Y^2 > 1/\lambda^2 \Rightarrow$ NO PROPAGATION AT THOSE FREQUENCIES.

9-23-75 (TUES)

CH. 4 (1,4,5,6) + 1 SPECIAL PROBLEM TO BE HANDED OUT THURS. DUE TUES 9-30

NOTES:

CH-4: FRESNEL AND FRAUNHOFER APPROXIMATION



HUYGEN'S-FRESNEL PRINCIPLE:

$$U_0(x_0, y_0) = \int_\Sigma\int U(x_1, y_1)\, \tilde{h}(x_0, y_0; x_1, y_1)\, dx_1\, dy_1$$

WHERE, FROM RAYLEIGH-SUMMERFIELD

$$h(x_0, y_0; x_1, y_1) = \frac{1}{j\lambda} \frac{e^{jr_{01}}}{j\lambda} \cos(\vec{n}, \vec{r}_{01})$$

ASSUMPTIONS:

1. $\cos(\vec{n}, \vec{r}_{01}) \cong 1$ ← OBLIQUITY FACTOR (GOOD WITHIN ~ 5%) ←

THIS IS A PARAXIAL TYPE OF APPROXIMATION

2. r_{01} IN DENOMINATOR $\cong z$

3. r_{01} IN EXPONENT

$$r_{01} = \left[z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2 \right]^{1/2}$$

MAKE A BINOMIAL EXPANSION:

$$\text{NOW: } \sqrt{1+b} = 1 + \tfrac{1}{2} b - \tfrac{1}{8} b^2 + \ldots$$

$$\simeq 1 + \tfrac{1}{2} b \quad \text{FOR } b \ll 1$$

$$\text{SINCE } r_{01} = z \sqrt{1 + \frac{(x - x_0)^2}{z^2} + \frac{(y - y_0)^2}{z^2}}$$

$$\simeq z \left[1 + \frac{1}{2} \frac{(x - x_0)^2}{z^2} + \frac{1}{2} \frac{(y - y_0)^2}{z^2} \right]$$

THIS IS THE FRESNEL APPROXIMATION GIVING THE FRESNEL TRANSFORM:

$$U(x_0, y_0) = \frac{1}{j\lambda z} \iint_{-\infty}^{\infty} e^{j \frac{2\pi}{\lambda} z \left[1 + \frac{(x_1 - x_0)^2}{2z^2} + \frac{(y_1 - y_0)^2}{2z^2} \right]} U(x_1, y_1) \, dx_1 dy_1$$

NOTE THAT IN THE FRESNEL APPROXIMATION, SPHERICAL WAVES ARE APPROXIMATED BY PARABOLIC WAVEFRONTS.

$$\text{NOW } U_0(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j \frac{k}{2z}(x_0^2 + y_0^2)}$$

$$\times \iint_{-\infty}^{\infty} U(x_1, y_1) \, e^{j \frac{k}{2z}(x_1^2 + y_1^2)} \, e^{-j \frac{2\pi}{\lambda z}(x_0 x_1 + y_0 y_1)} \, dx_1 dy_1$$

$$\text{IF } f_x \triangleq \frac{x_0}{\lambda z}, \quad f_Y \left(\frac{y_0}{\lambda z} \right)$$

$$\text{THEN } U_0\left(\frac{x_0}{\lambda z}, \frac{y_0}{\lambda z}\right) = U_0(f_x, f_Y)$$

$$= \frac{e^{jkz}}{j\lambda z} e^{j \frac{k}{2z}(x_0^2 + y_0^2)} \mathcal{F} \left[U_1(x_1, y_1) \, e^{j \frac{k}{2z}(x_1^2 + y_1^2)} \right]$$

FOR FRESNEL APPROXIMATION TO BE VALID, $z^3 \gg \frac{\pi}{4\lambda} \left[(x_0 - x_1)^2 + (y_0 - y_1)^2 \right]^2_{MAX}$

FOR FRAUNHOFER APPROXIMATION TO BE VALID, $z \gg \frac{k}{2}(x_1^2+y_1^2)_{MAX}$

THESE ARE TRUE AS FAR AS THE INTEGRAL'S KERNAL IS CONCERNED. HOWEVER, EVALUATION OF THE INTEGRAL DOES NOT DICTATE SUCH SEVERE CONSTRAINT, SINCE THE PHASE TERMS OSCILLATE SO MUCH IN BOTH TRUE AND APPROXIMATED VERSIONS. CONCEPT CALLED;

METHOD OF STATIONARY PHASE

PROBLEM: FIND AN ASYMPTOTIC EXPRESSION FOR

$$I = \int_{-\infty}^{\infty}\!\!\int g(x,y)\, e^{jk\mu(x,y)}\, dx\, dy$$

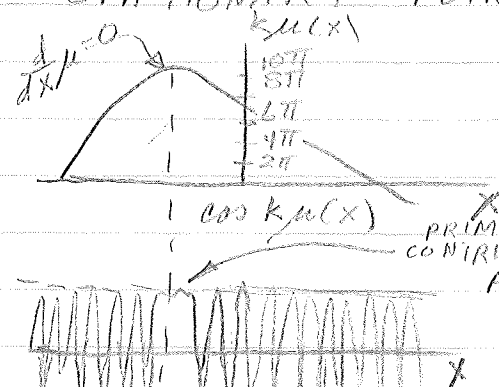
FOR $k \gg 1$ (ie VERY LARGE k)

PHYSICAL ARUMENT: CONSIDER THE 1-D INTEGRAL

$$\int g(x) e^{jk\mu(x)} dx$$

$$= \int g(x) \cos k\mu(x) - j \int g(x) \sin k\mu(x) dx$$

FOR LARGE k, THE INTEGRAND OSCILLATES RAPIDLY EXCEPT AT SO-CALLED "STATIONARY POINTS" WHERE $\frac{d}{dx}\mu(x) = 0$



PRIMARY CONTRIBUTION TO INTEGRAL LIES AT STATIONARY PHASE

(CONT)

(SEE APPENDIX IN BORN & WOLF ON STATIONARY PHASE.)

FOR TWO DIMENSIONS

LET X_0, Y_0 BE THE ONLY STATIONARY POINT OF $\mu(X,Y)$. THAT IS $\frac{\delta U}{\delta X}\Big|_{X_0,Y_0} = \frac{\delta U}{\delta Y}\Big|_{X_0,Y_0} = 0$. FURTHER ASSUMTIONS SATISFIED IS THE CASE OF INTEREST:

$$\alpha_0 \triangleq \frac{\delta^2 U}{\delta X^2}\Big|_{X_0,Y_0} \neq 0 \qquad \beta_0 \triangleq \frac{\delta^2 U}{\delta Y^2}\Big|_{X_0,Y_0} \neq 0$$

FURTHERMORE, $\alpha_0\beta_0 - \gamma_0^2 \neq 0$ WHERE $\gamma_0 \triangleq \frac{\delta^2 U}{\delta X \delta Y}\Big|_{X_0 Y_0}$.

EXPAND $\mu(X,Y)$ ABOUT THE STATIONARY PT, IN A TWO-DIMENSIONAL TAYLOR SERIES

$$U(X,Y) = \mu(X_0 Y_0) + \frac{1}{2}\alpha_0(X-X_0)^2 + \frac{1}{2}\beta_0(Y-Y_0)^2 + \gamma(X-X_0)(Y-Y_0) + \ldots$$

CONCENTRATE ON THE LOWEST ORDER TERMS DUE TO THE FACT THAT THE MAIN CONTRIBUTIONS TO I WILL COME FROM $X \simeq X_0$ AND $Y \simeq Y_0$. BASED ON THE ABOVE, WE REPLACE $g(X,Y)$ IN I BY $g(X_0,Y_0)$. GIVES:

$$I \simeq g(X_0,Y_0)\, e^{jk\mu(X_0,Y_0)} \times \int_{-\infty}^{\infty}\!\!\int e^{j\frac{k}{2}(\alpha_0\xi^2 + \beta_0\eta^2 + 2\gamma_0\xi\eta)}\, d\xi\, d\eta$$

WHERE WE'VE LET $\xi = X - X_0$, $\eta = Y - Y_0$.

FROM INTEGRAL TABLES:

$$\int_{-\infty}^{\infty}\!\!\int e^{j\frac{k}{2}(\alpha_0\xi^2 + \beta_0\eta^2 + 2\gamma_0\xi\eta)}\, d\xi\, d\eta = \frac{2\pi j\sigma}{\sqrt{|\alpha_0\beta_0 - \gamma_0^2|}} \cdot \frac{1}{K}$$

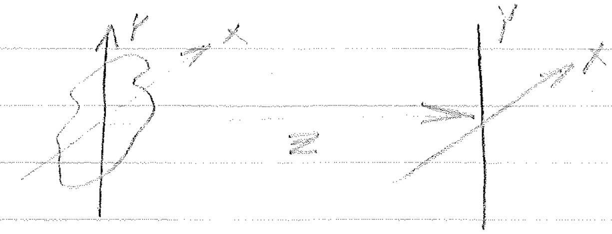
$$\text{WHERE} \quad \sigma = \begin{cases} 1, & \alpha_0\beta_0 > \gamma_0^2 \; ; \; \alpha_0 > 0 \\ -1, & \alpha_0\beta_0 > \gamma_0^2 \; , \; \alpha_0 < 0 \\ -j, & \alpha_0\beta_0 < \gamma_0^2 \end{cases}$$

THUS

$$\int_{-\infty}^{\infty}\!\!\int g(X,Y)\, e^{jk\mu(X,Y)}\, dX\, dY \simeq \frac{2\pi j\sigma}{\sqrt{|\alpha_0\beta_0 - \gamma_0^2|}}\, g(X_0,Y_0)\, \frac{e^{jk\mu(X_0,Y_0)}}{K}$$

9-25-75 (THURS.)

APPLICATION: FRESNEL DIFFRACTION DEEP IN NEAR FIELD



FRESNEL APPROXIMATION (TRANSFORM)

$$\tilde{U}(x_0,y_0) = \frac{e^{jkz}}{j\lambda z}\iint_{-\infty}^{\infty} t(x,y)\, e^{j\frac{\pi}{\lambda z}[(x-x_0)^2+(y-y_0)^2]}\, dx\, dy$$

LET $k = \frac{\pi}{\lambda z}$, $\mu(x,y) = (x-x_0)^2+(y-y_0)^2$

$\frac{\delta \mu}{\delta x} = 2(x-x_0)$; $\frac{\delta \mu}{\delta y} = 2(y-y_0)$

$\Rightarrow \frac{\delta \mu}{\delta x}\Big|_{x=x_0} = 0$; $\frac{\delta \mu}{\delta y}\Big|_{y=y_0} = 0$

$\Rightarrow (x_0,y_0)$ ARE THE STATIONARY POINTS

EVALUATE @ $(x,y) = (x_0,y_0)$:

$\alpha_0 = \frac{\delta^2 \mu}{\delta x^2}\Big|_{x_0,y_0} = 2$

$\beta_0 = \frac{\delta^2 \mu}{\delta y^2}\Big|_{x_0,y_0} = 2$

$\gamma_0 = \frac{\delta^2 \mu}{\delta x \delta y} = 0$

NOW, SINCE $\alpha_0\beta_0 = 4 > \gamma_0^2 = 0 \Rightarrow \sigma = \underline{1}$

∴ USING THE TABULATED INTEGRAL:

$$U(x_0,y_0) = \frac{e^{jkz}}{j\lambda z}\, t(x_0,y_0)\, \frac{2\pi j}{\sqrt{4}}\, e^{jk0}\, \frac{1}{\pi/\lambda z}$$

RECALL:

$$I = \iint g(x,y)\, e^{jk\mu(x,y)}\, dx\, dy$$

LET $\xi = x - x_0$, $\eta = y - y_0$

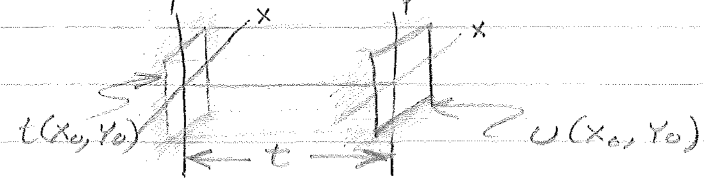
USING TAYLOR SERIES EXPANSION:

$$I \simeq g(x_0,y_0)\, e^{jk\mu(x_0,y_0)} \times \iint e^{j\frac{k}{2}(\alpha\xi^2 + \beta_0\eta^2 + 2\gamma_0\xi\eta)}\, d\xi\, d\eta$$

∴ e^{j0} CAME FROM $e^{j\frac{\pi}{\lambda z}[(x-x_0)^2+(y-y_0)^2]}\Big|_{x_0 y_0} = e^{j0}$

ANYWAY, WE WIND UP WITH

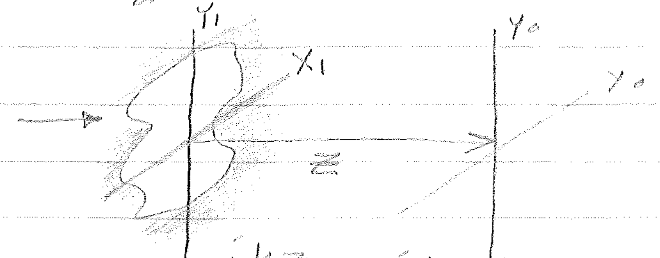
$$U(x_0, y_0) = t(x_0, y_0)\, e^{jkz}$$ ← PROJECTION OF APERTURE BASED ON GEOMETRIC OPTICS



RECALL FRAUNHOFER APPROXIMATION:

$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x_0^2 + y_0^2)}$$

$$\times \iint_{-\infty}^{\infty} U(x_1, y_1)\, e^{-j\frac{2\pi}{\lambda z}(x_1 x_0 + y_1 y_0)}\, dx_1\, dy_1$$



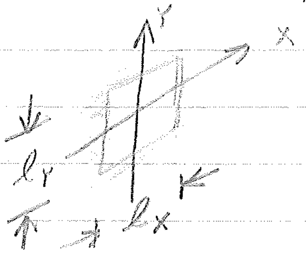
$$U(x_0, y_0) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2\pi}(x_0^2 + y_0^2)}\, \mathcal{F}\left[U(x_1, y_1)\right]\Big|_{\substack{f_x = x_0/\lambda z \\ f_y = y/\lambda z}}$$

IF SCREEN PLACED IN x_0, y_0 PLANE, WE "SEE" INTENSITY:

$$I(x_0, y_0) \triangleq |U(x_0, y_0)|^2$$

$$= \frac{1}{\lambda^2 z^2} \left|\mathcal{F}\{U(x_0, y_0)\}\right|^2_{\substack{f_x = \frac{x}{\lambda z} \\ f_y = \frac{y}{\lambda z}}}$$

I. CONSIDER: SQUARE APERTURE



$U_{TRANS} = U_{INCIDENCE} \cdot t(X,Y)$

NOW $t(X,Y) = \mathrm{rect}\left(\frac{X}{l_X}, \frac{Y}{l_Y}\right)$

LET $U_{INCIDENCE} = \underline{1}$

$\Rightarrow U_{TRANS} = t(X,Y)$

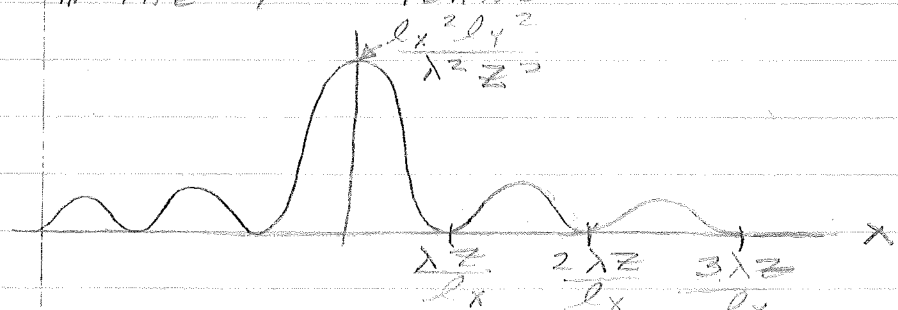
THEN IN FRAUNHOFER REGION:

$$|U(X,Y)|^2 = \frac{1}{\lambda^2 Z^2}\left|\mathcal{F}\{t(x,y)\}\right|^2_{\substack{f_X = X/\lambda Z \\ f_Y = Y/\lambda Z}}$$

$$\mathcal{F}\left[\mathrm{rect}\left(\frac{X}{l_X}, \frac{Y}{l_Y}\right)\right] = l_X l_Y \,\mathrm{sinc}(l f_X, l f_Y)$$

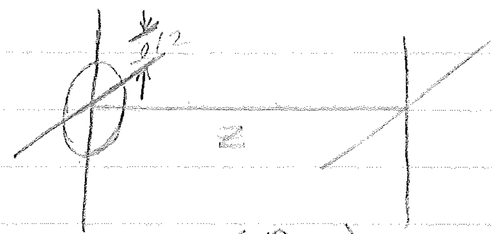
$$I(X,Y) = |U(X,Y)|^2 = \frac{l_X^2 l_Y^2}{\lambda^2 Z^2}\,\mathrm{sinc}^2\left(\frac{l_X X}{\lambda Z}\right)\mathrm{sinc}^2\left(\frac{l_Y Y}{\lambda Z}\right)$$

IN THE $Y=0$ PLANE



HALF WIDTH OF THE MAIN LOBE IS $\frac{\lambda Z}{l_X}$

II. CONSIDER CIRCULAR APERTURE



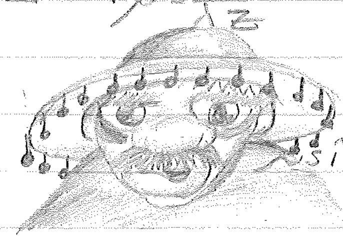
$$t(r_1) = \mathrm{circ}\left(\frac{r_1}{l/2}\right)$$

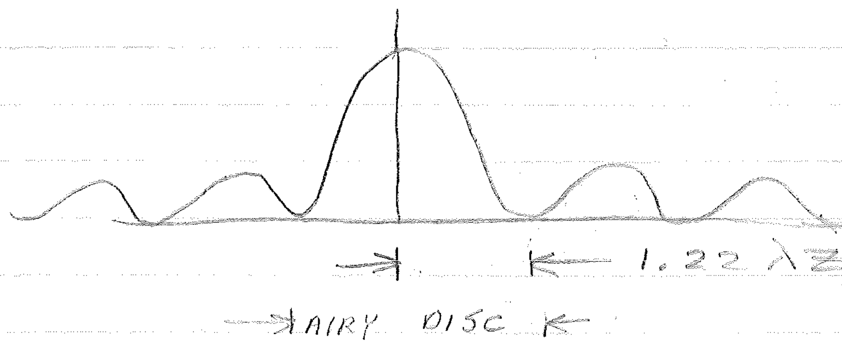
$$I(r_0) = \frac{1}{\lambda^2 z^2}\left|\mathcal{B}\left\{\mathrm{circ}\left(\frac{r_1}{l/2}\right)\right\}\right|^2\Bigg|_{\rho = \frac{r_0}{\lambda z}}$$

$$\mathcal{B}\left[\mathrm{circ}\left(\frac{r_1}{l/2}\right)\right] = \left(\frac{l}{2}\right)^2 \frac{J_1(\pi l \rho)}{l\rho/2}$$

$$\Rightarrow I(r_0) = \frac{1}{\lambda^2 z^2}\left(\frac{l}{2}\right)^4 \left|\frac{J_1(\pi l \rho)}{l\rho/2}\right|^2\Bigg|_{\rho = \frac{r_0}{\lambda z}}$$

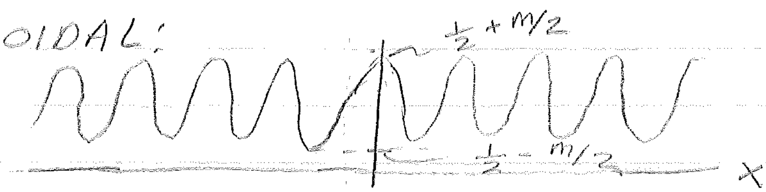
= SQUARE OF SOMBRARO FUNCTION



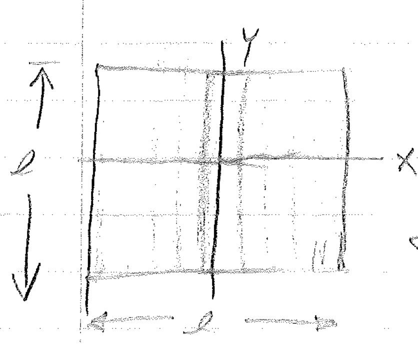


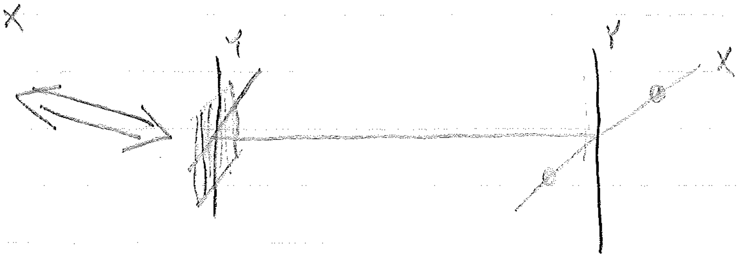
III. CONSIDER GRATINGS

SINUSOIDAL:



$$t(x,y) = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 x)\right]\mathrm{rect}\left(\frac{x}{l}, \frac{y}{l}\right)$$





(CONT →)

$$I(x,Y) = \frac{1}{\lambda^2 z^2}\left|\mathcal{F}\{t(x,Y)\}\right|^2 \Big|_{f_x = x/\lambda f,\ f_Y = Y/\lambda f}$$

$$t(x,Y) = \left[\tfrac{1}{2} + \tfrac{m}{2}\cos(2\pi f_0 x)\right] \mathrm{rect}\left[\tfrac{x}{l}, \tfrac{Y}{l}\right]$$

$$\mathcal{F}[t(x,Y)] = \mathcal{F}\left[\tfrac{1}{2} + \tfrac{m}{2}\cos(2\pi f_0 x)\right] * \mathcal{F}\left[\mathrm{rect}\left(\tfrac{x}{l}, \tfrac{Y}{l}\right)\right]$$

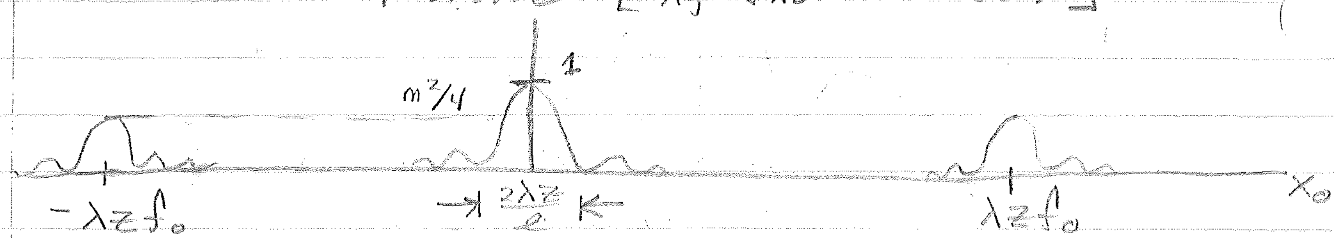
$$\begin{cases} \mathcal{F}\left[\mathrm{rect}\left(\tfrac{x}{l}, \tfrac{Y}{l}\right)\right] = l^2\,\mathrm{sinc}(l f_x, l f_Y) \\ \mathcal{F}\left[\tfrac{1}{2}\right] = \tfrac{1}{2}\,\delta(f_x, f_Y) \\ \mathcal{F}\left[\tfrac{m}{2}\cos(2\pi f_0 x)\right] = \tfrac{m}{4}\,\delta(f_x + f_0, f_Y) + \tfrac{m}{4}\,\delta(f_x - f_0, f_Y) \end{cases}$$

$$\mathcal{F}[t(x,Y)] = \left[\tfrac{1}{2}\delta(f_x,f_Y) + \tfrac{m}{4}\delta(f_x+f_0,f_Y) + \tfrac{m}{4}\delta(f_x-f_0,f_Y)\right] \circledast l^2\,\mathrm{sinc}(l f_x, l f_Y)$$

$$= \frac{l^2}{2}\,\mathrm{sinc}(l f_x, l f_Y) + \frac{m l^2}{4}\,\mathrm{sinc}\left[l(f_x+f_0), l f_Y\right] + \frac{m l^2}{4}\,\mathrm{sinc}\left[l(f_x - f_0), l f_Y\right]$$

NEGLECTING CROSS-TERMS:

$$I(x_0,Y_0) = \frac{l^2}{2\lambda z}\,\mathrm{sinc}^2\left(\frac{l Y_0}{\lambda z}\right)\left[\mathrm{sinc}^2 \frac{l x_0}{\lambda z} + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{l}{\lambda z}(x_0 + \lambda z f_Y)\right] + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{l}{\lambda f}(x_0 - \lambda z f_0)\right]\right]$$



-1^{ST} ORDER $\qquad$ 0^{TH} ORDER $\qquad$ 1^{ST} ORDER

$\Rightarrow$ FOR GOOD RESOLUTION, $f_0 \gg 2/l$

ie $l \gg 2/f_0$ $\Rightarrow$ LOTS OF PERIODS OF GRATING IN l.

10-2-75 (THURS)

GRATING 3

1. THIN AMPLITUDE GRATING

$d \ll 1/f_0$; d: THICKNESS

f_0 = GRATING FREQ

$$t(x,Y) = \left[\tfrac{1}{2} + \tfrac{m}{2}\cos 2\pi f_0 x\right] \mathrm{rect}\left[\tfrac{x}{\ell}, \tfrac{Y}{\ell}\right]$$

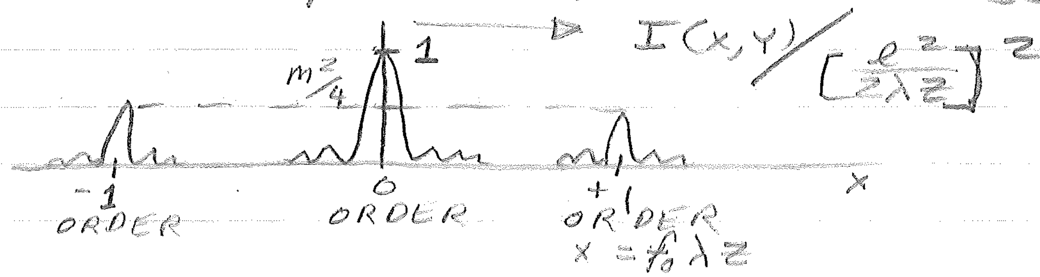
$$\mathcal{F}[t(x,Y)] = \left[\tfrac{1}{2}\delta(f_x,f_Y) + \tfrac{m}{4}\delta(f_x - f_0, f_Y) + \tfrac{m}{4}\delta(f_x + f_0, f_Y)\right] * \ell^2 \mathrm{sinc}[\ell f_x, \ell f_Y]$$

$$= \frac{\ell^2}{2}\mathrm{sinc}[\ell f_x, \ell f_Y] + \frac{\ell^2 m}{4}\mathrm{sinc}[f_x - f_0, f_Y] + \frac{\ell^2 m}{4}\mathrm{sinc}[f_x + f_0, f_Y]$$

$$U(x_0,Y_0) = \frac{1}{j\lambda z} e^{jkz} e^{j\frac{k}{2z}(x_0^2 + Y_0^2)} \times \left\{ \frac{\ell^2}{2}\mathrm{sinc}\left(\frac{\ell Y_0}{\lambda z}\right)\left[\mathrm{sinc}\,\frac{\ell x_0}{\lambda z} + \frac{m}{2}\mathrm{sinc}\left[\frac{\ell}{\lambda z}(x_0 + f_0\lambda z)\right] + \frac{m}{2}\mathrm{sinc}\left[\frac{\ell}{\lambda z}(x_0 - f_0\lambda z)\right]\right]\right\}$$

$$I(x_0,Y_0) = |U(x_0,Y_0)|^2 \quad \text{FOR } f_0 \gg 2/\ell$$

$$\cong \left[\frac{\ell^2}{2\lambda z}\right]^2 \mathrm{sinc}^2\left(\frac{\ell Y_0}{\lambda z}\right) \times \left[\mathrm{sinc}^2\frac{\ell x_0}{\lambda z} + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{\ell}{\lambda z}(x_0 + f_0\lambda z)\right] + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{\ell}{\lambda z}(x_0 - f_0\lambda z)\right]\right]$$



$\eta \triangleq$ DIFFRACTION EFFICIENCY

$= m^2/16 \Rightarrow \eta_{MAX} = 1/16 = 6.25\%$

FOR INFINATE GRATING ($\ell = \infty$)

$$t(x,Y) = \frac{1}{2} + \frac{m}{4}e^{j2\pi f_0 x} + \frac{m}{4}e^{-j2\pi f_0 x}$$

2. <u>THIN</u> PHASE GRATING (Pg 69 IN GOODMAN)

$d << 1/f_o$

$$t(x,y) = \left[e^{j\frac{m}{2}\sin(2\pi f_o x)}\right] \mathrm{rect}\left[\frac{x}{\ell}, \frac{y}{\ell}\right]$$

$$= \left[\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) e^{j2\pi q f_o x}\right] \mathrm{rect}\left[\frac{x}{\ell}, \frac{y}{\ell}\right]$$

$$\mathcal{F}[t(x,y)] = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \delta(f_x - q f_o, f_y)$$

$$* \; \ell^2 \,\mathrm{sinc}(\ell f_x, \ell f_y)$$

$$= \ell^2 \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \mathrm{sinc}\left[\ell\{f_x - q f_o\}, \ell f_y\right]$$

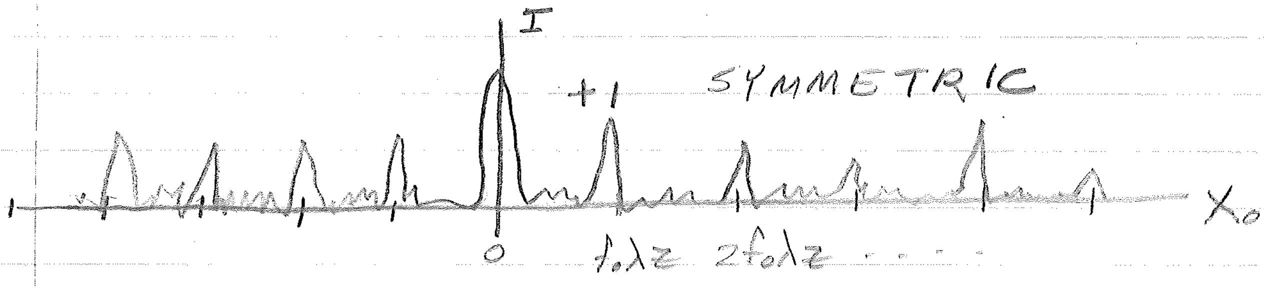
$$U(x_o, y_o) = \frac{\ell^2}{j\lambda z} e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}$$

$$\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \mathrm{sinc}\left[\ell\left(\frac{x_o}{\lambda z} - q f_o\right), \ell \frac{y_o}{\lambda z}\right]$$

$$I(x_o, y_o) = |U(x,y)|^2$$

$$\simeq \left(\frac{\ell^2}{\lambda z}\right)^2 \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right) \mathrm{sinc}^2\left[\frac{\ell}{z\lambda}(x_o - q f_o \lambda z)\right]$$

$$\times \,\mathrm{sinc}^2\left[\frac{\ell y_o}{\lambda z}\right]$$



FOR LARGE APERTURE

$\eta = J_1^2\left(\frac{m}{2}\right)$ = DIFFRACTION EFFIC.

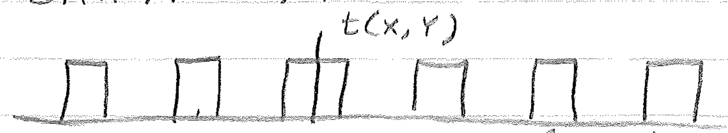
$\eta_{MAX} \simeq J_1^2(1.842) = (.5819)^2 = 33.9\%$

FOR $m = 3.684$

$t(x,y) = \sum J_q\left(\frac{m}{2}\right) e^{j2\pi f_o x} \Rightarrow J_1^2\left(\frac{m}{2}\right)$

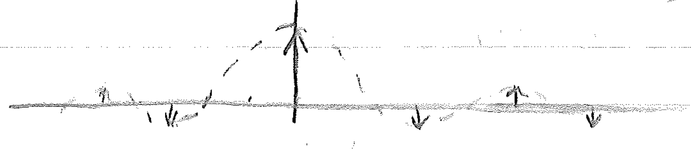
THICK GRATING MAY HAVE LARGER η

NOTE: SQUARE WAVE GRATING (AMPLITUDE GRATING):



$t \sim \mathrm{rect}(x) * \mathrm{comb}(x)$

$\mathcal{F}[t] = \mathrm{sinc}(f_x)\,\mathrm{comb}(f_x)$



IT TURNS OUT $\eta = 10.1\%$

10-6-75 (TUES)

H.W #4 - DUE TUES 10/14

CH. 5 (2, 4, 5, 8, 9, 10)

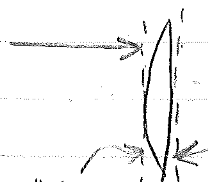
QUIZ - TUES 10/21

NOTES: (CHAPT. 5)

"THIN" LENS ⇒ RAY EMERGES AT SAME COORDINATE IT ENTER

(NO TRANSLATION)

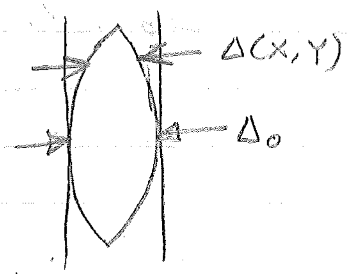
(ALSO, NO ATTENUATION)



$U_\ell'(x,Y) = t(x,Y)\,U_\ell(x,Y)$

NON ATTENUATING ⇒ $t(x,Y)$ = PURE PHASE = $e^{j\phi(x,Y)}$

$$\phi(x,Y) = \left(\frac{2\pi}{\lambda} n\right)\Delta(x,Y) + \underbrace{\frac{2\pi}{\lambda}\left[\Delta_0 - \Delta(x,Y)\right]}_{\text{DUE TO AIR}}$$



n = REFRACTIVE INDEX

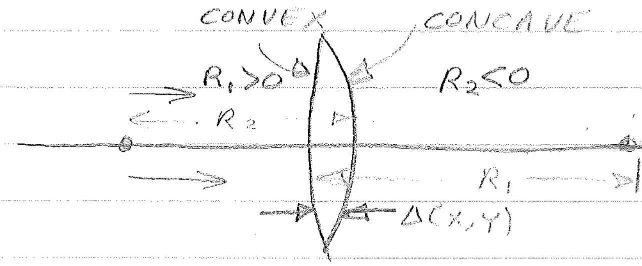
$\simeq 1.5$ FOR OPTICAL GLASS

$$\phi(x,Y) = k\Delta_0 + k(n-1)\Delta(x,Y)$$

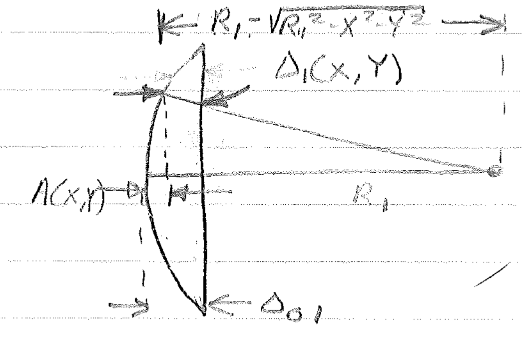
$k\Delta_0$ = PHASE DELAY IN FREE SPACE IN ABSENCE OF LENS

$k(n-1)\Delta(x,Y)$ = EXCESS DELAY

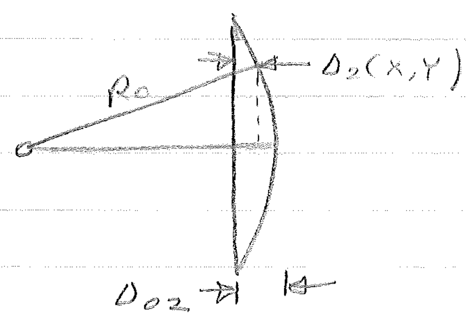
SIGN CONVENTION



AS RAYS TRAVEL FROM LEFT TO RIGHT, EACH CONVEX SPHERICAL SURFACE ENCOUNTERED IS TAKEN TO HAVE A POSITIVE RADIUS OF CURVATURE, AND EACH CONCAVE SPHERICAL SURFACE ENCOUNTERED IS TAKEN TO HAVE A NEGATIVE RADIUS OF CURVATURE.



$$\Delta_1(X,Y) = \Delta_{01} - \Lambda(X,Y)$$
$$= \Delta_{01} - \left[R_1 - \sqrt{R_1^2 - X^2 - Y^2}\right]$$
$$= \Delta_{01} - R_1\left[1 - \sqrt{1 - \frac{(X^2+Y^2)}{R_1^2}}\right]$$



$$\Delta_2(X,Y) = \Delta_{02} + \left(R_2 + \sqrt{R_2^2 - X^2 - Y^2}\right)$$
$$= \Delta_{02} + R_2\left[1 - \sqrt{1 - \frac{X^2+Y^2}{R_2^2}}\right]$$

PARAXIAL APPROXIMATION IS NOW APPLIED

$$\sqrt{1-\frac{x^2+y^2}{R_1^2}} \simeq 1-\frac{x^2+y^2}{2R_1^2} \quad ; x^2+y^2 << R_1^2$$
$$\sqrt{1-\frac{x^2+y^2}{R_2^2}} \simeq 1-\frac{x^2+y^2}{2R_2^2} \quad ; x^2+y^2 << R_2^2$$

$$\Delta(x,y) = \Delta_1(x,y) + \Delta_2(x,y)$$
$$= \Delta_0 - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

RECALL

$$t_\ell(x,y) = e^{jk\Delta_0}\, e^{jk(n-1)\Delta(x,y)}$$
$$= e^{jk\Delta_0}\, e^{jk(n-1)\left[\Delta_0 - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]}$$
$$= e^{jkn\Delta_0}\, e^{-jk(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}$$

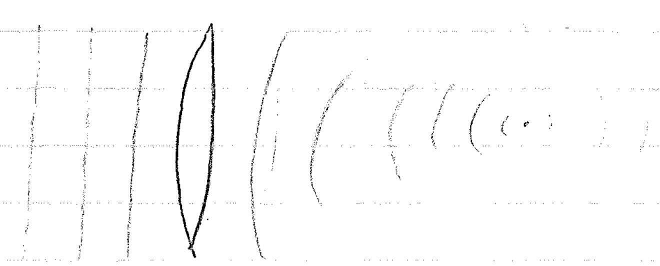
FROM CLASSICAL OPTICS, THE LENS' FOCAL LENGTH, f, IS DEFINED AS

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1}-\frac{1}{R_2}\right]$$

THUS

$$t_\ell(x,y) = e^{jkn\Delta_0}\, e^{-j\frac{k}{2f}(x^2+y^2)}$$

MAY DROP $e^{jkn\Delta_0}$. WE DON'T CONCERN OURSELVES WITH THIS POSITION INVARIANT PHASE DELAY.

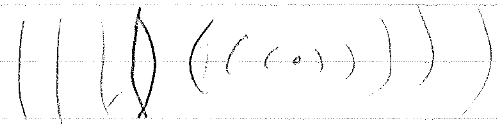


$\hat{t}(x,y) = e^{-j\frac{k}{2f}(x^2+y^2)}$ WILL MAP AN INCIDENT PLANE WAVE TO A FOCUS POINT IN REAR FOCAL PLANE (CONVERGING SPHERICAL WAVE)

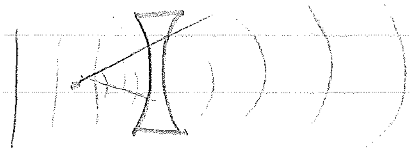
"POSITIVE" LENSES $\Rightarrow f > 0$

"NEGATIVE" LENSES $\Rightarrow f < 0$

POSITIVE:



NEGATIVE:

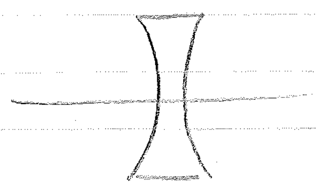


DOUBLE CONVEX LENS:

$$\Rightarrow \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$R_1 > 0,\ R_2 < 0,\ n > 1 \Rightarrow f > 0$

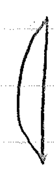
DOUBLE CONCAVE LENS:



$R_1 < 0,\ R_2 > 0,\ n > 1$

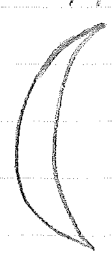
$\Rightarrow f < 0$

PLANO - CONVEX



$R_1 > 0\ ,\ R_2 = \infty \Rightarrow f > 0$

POSITIVE MENISCUS $\Rightarrow$



$R_1 > 0\ R_2 > 0$

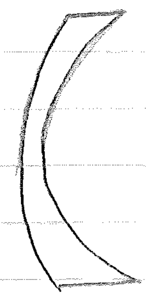
$R_2 > R_1 \Rightarrow f > 0$

PLANO CONCAVE

$R_1 < 0\ ,\ R_2 = \infty$

$\Rightarrow f < 0$

NEGATIVE MENISCUS



$R_1 > 0, R_2 > 0$

$R_1 > R_2$

$f < 0$

RECALL FOR A DIVERGING SPHERICAL WAVE:

$$\frac{e^{jkr_{01}}}{r_{01}}$$ (Pg. 38)

$$r_{01} \approx z\left(1 + \frac{x^2+y^2}{2z^2}\right)$$

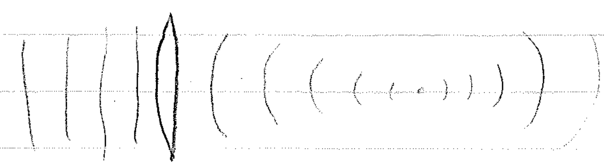
$$\Rightarrow \frac{e^{jkr_{01}}}{r_{01}} = \frac{e^{jkz}\, e^{j\frac{\pi}{\lambda z}(x^2+y^2)}}{z}$$

$$\overset{z=-f}{\Rightarrow} \frac{e^{jkr_{01}}}{r_{01}} \sim e^{-j\frac{\pi}{\lambda f}(x^2+y^2)}$$

10-9-74 (THURS.)

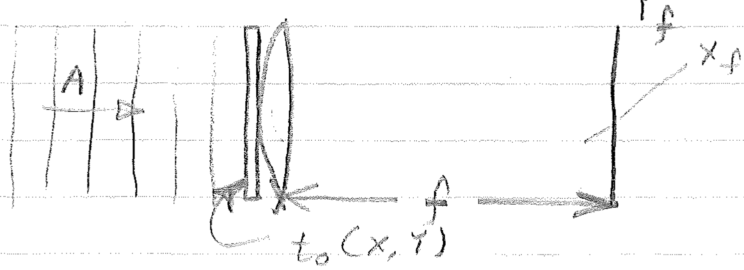
HOMEWORK #4 DUE NEXT THURS (10-16)

AMPLITUDE TRANSMITTANCE OF THIN LENS



$$t_l(x,Y) = e^{-j\frac{\pi}{\lambda f}(x^2+Y^2)} = e^{-j\frac{k}{2f}(x^2+Y^2)}$$

TRANSPARENCY IN FRONT OF LENS



TO THE RIGHT OF THE LENS WE GOT

$$U_r(x,Y) = A\,t_o(x,Y)\,e^{-j\frac{k}{2f}(x^2+Y^2)}\;P(x,Y)$$

$\ni P(x,Y)$ = THE PUPIL FUNCTION

$$= \begin{cases} 1 & ; \text{ WITHIN THE APERTURE} \\ 0 & ; \text{ OUTSIDE THE APERTURE} \end{cases}$$

ASSUME f IS LARGE ENOUGH TO USE FRESNEL DIFFRACTION.

FOR FRESNEL DIFFRACTION (WITH $z = f$)

$$\tilde{U}(x_f, y_f) = \frac{e^{jkf}}{j\lambda f}\, e^{j\frac{k}{2f}(x_f^2+y_f^2)}$$

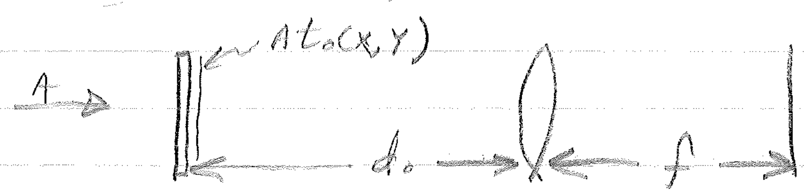
$$\int\!\!\int_{-\infty}^{\infty} \underbrace{U_r(x,Y)}_{A\,t_o(x,Y)\,e^{-j\frac{\pi}{\lambda f}(x^2+Y^2)}\,P(x,Y)} e^{j\frac{k}{2f}(x^2+Y^2)}\, e^{-j\frac{2\pi}{\lambda f}(x_f x + Y_f Y)}\, dx\,dY$$

OR
$$U(x_f, y_f) = \frac{A e^{jkf}}{j\lambda f} e^{j\frac{k}{2f}(x_f^2+y_f^2)}$$
$$\times \int_{-\infty}^{\infty}\!\!\int t_o(x,y)\, P(x,y)\, e^{-j\frac{2\pi}{\lambda f}(x_f x + y_f y)}\, dx\, dy$$
$$= \frac{A e^{jkf}}{j\lambda f} e^{j\frac{k}{2f}(x_f^2+y_f^2)} \mathcal{F}\left[t_o(x,y)\, P(x,y)\right]\Big|_{\substack{f_x = x/\lambda f \\ f_y = y/\lambda f}}$$

$$I(x_f, y_f) = |U(x_f, y_f)|^2$$
$$= \left(\frac{A}{\lambda f}\right)^2 \left|\mathcal{F}\left[t_o(x,y) P(x,y)\right]\right|^2_{\substack{f_x = x/\lambda f \\ f_y = y/\lambda f}} \Leftarrow \text{POWER SPECTRUM}$$

IF LENS COMPLETELY COVERS TRANSPARENCY, REPLACE $P(x,y) = 1$

CONSIDER NOW



Eq. 4-11, Pg 60: TRANSFER FNCTION FOR FRESNEL DIFF:
$$\tilde{H}(f_x, f_y) = e^{jkz} e^{-j\pi\lambda z(f_x^2+f_y^2)}$$
$$\mathcal{F}\{A\, t(x_o, y_o)\} = F_o(f_x, f_y)$$

LIGHT BEFORE THE LENS:
$$F_l(f_x, f_y) = F_o(f_x, f_y)\, e^{-j\pi\lambda d(f_x^2+f_y^2)}$$
$$U_f(x_f, y_f) = \frac{A e^{jkf}}{j\lambda f} e^{j\frac{k}{2f}(x_f^2+y_f^2)} F_l(f_x, f_y)$$
$$= \frac{A}{j\lambda f} e^{j\frac{k}{2f}\left(1-\frac{d_o}{f}\right)(x_f^2+y_f^2)} \mathcal{F}\left[t_o(x,y)\right]\Big|_{\substack{f_x = x_f/\lambda f \\ f_y = y_f/\lambda f}}$$

THUS, FOR $d = f$, WE GOTTA EXACT FOURIER TRANSFORM ON D BACK FOCAL PLANE

$$\tilde{U}_f(x_f, y_f) = \frac{A}{j\lambda f} \mathcal{F}\left[t_o(x,y)\right]\Big|_{\substack{f_x = x_f/\lambda f \\ f_y = y_f/\lambda f}}$$

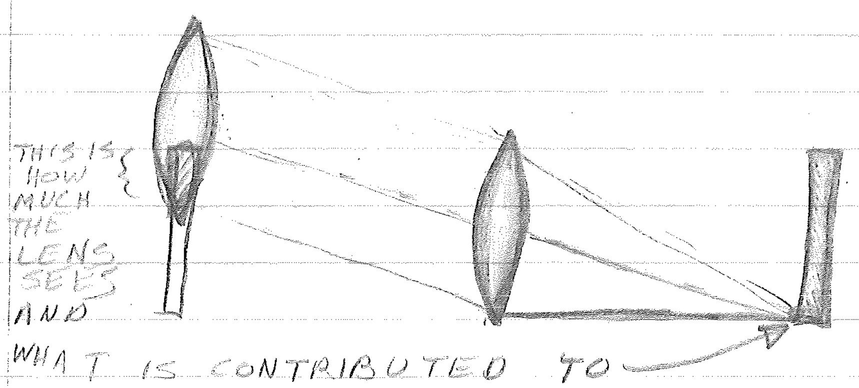
FOR VERY LARGE d

$$U_f(x_f, y_f) = \frac{A}{j\lambda f} e^{j\frac{k}{2f}\left(1-\frac{d_o}{f}\right)(x_f^2 + y_f^2)}$$

$$\cdot \iint_{-\infty}^{\infty} t_o(x_o, y_o)\, P\left(x_o + \frac{d_o}{f} x_f,\ y_o + \frac{d_o}{f} y_f\right)$$

$$\times\, e^{-j\frac{2\pi}{\lambda f}(x_o x_f + y_o y_f)}\, dx_o\, dy_o$$

VIGNETTING: LOOSE HIGH FREQUENCIES



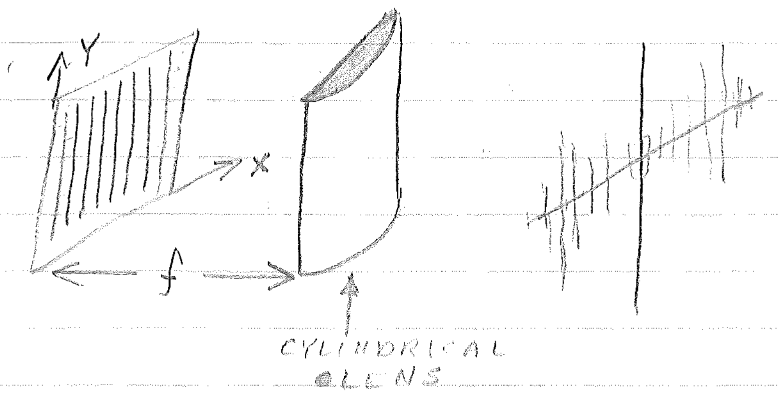
TO GET AROUND IT

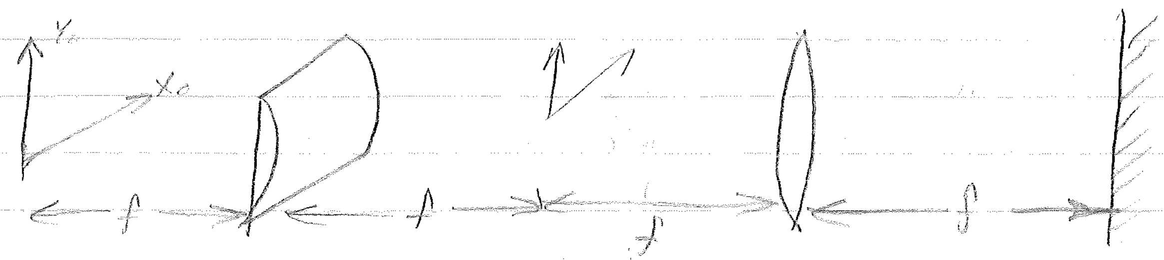
1. USE A BIGGER LENS
2. SMALLER OBJECT TRANSPARANCY
3. LENS WITH A SMALLER FOCAL LENGTH
4. USE SMALLER λ
5. BRING TRANSPARANCY AGAINST THE LENS, OR AT LEAST CLOSER

10-14-75 (TUES)

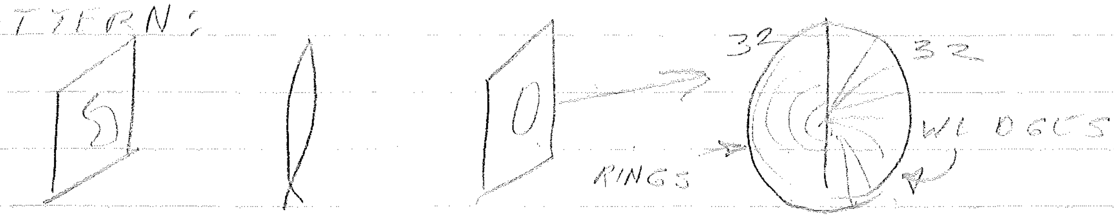
QUIZ TUES THRU CHAPT. 5

1-D FOURIER TRANSFORM w/ CYLYNDRICAL LENS





PATTERN:



GET 64 PATTERN VECTOR.

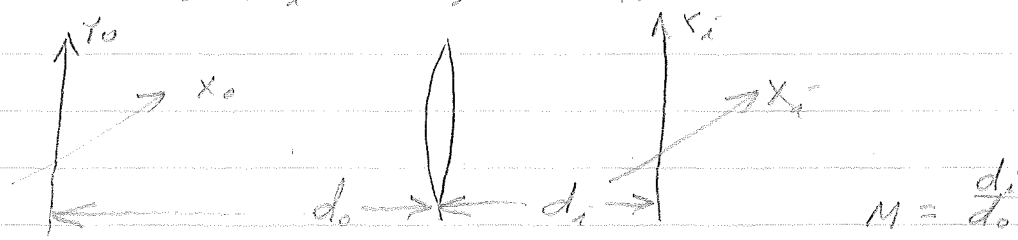
IMAGE FORMATION IN MONOCHROMATIC LITE

FROM HUYGEN'S FRESNEL:

$$\tilde{U}_i(x_i, y_i) = \int_{-\infty}^{\infty}\int U_o(x_o, y_o)\, \tilde{h}(x_i, y_i; x_o, y_o)\, dx_o\, dy_o$$

FOR IDEAL CASE, WE WANT

$$\tilde{h} \triangleq \tilde{K}\, \delta(x_i \pm M x_o, y_i \pm M y_o)$$



GOODMAN COMES UP WITH:

$$\tilde{h}(x_i, y_i; x_o, y_o) = \frac{1}{\lambda^2 d_o d_i} e^{j\frac{k}{2d_i}(x_i^2+y_i^2)} e^{j\frac{k}{2d_o}(x_o^2+y_o^2)}$$

$$\times \int_{-\infty}^{\infty}\int P(x,y)\, e^{j\frac{k}{2}\left(\frac{1}{d_o}+\frac{1}{d_i}-\frac{1}{f}\right)(x^2+y^2)}$$

$$\times e^{-jk\left[\left(\frac{x_o}{d_o}+\frac{x_i}{d_i}\right)x + \left(\frac{y_o}{d_o}+\frac{y_i}{d_i}\right)y\right]} dx\, dy$$

FIRST TERM IN INTEGR. VANISHES WHEN

$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$; LENS LAW

LET $e^{k/2d_o(x_o^2+y_o^2)} \approx e^{j \frac{k}{2d_o}\left(\frac{x_i^2+y_i^2}{M^2}\right)}$

CANCEL ALL COMPLEX COMPLEXITIES GIVES

$$\tilde{h}(x_i, y_i, x_o, y_o) = \frac{1}{\lambda^2 d_o d_i} \int_{-\infty}^{\infty}\int P(x,y)\, e^{-j\frac{2\pi}{\lambda d_i}[(x_i + M x_o)x + (y_i + M y_o)y]} dy\, dx$$

= SHIFTED FRAUNHOFER DIFFRACTION PATTERN OF THE PUPIL FUNCTION.

LET $\lambda \to 0$. WE GET GEOMETRICAL OPTICS RESULTS:

$$h(x_i, y_i; x_o, y_o) = \frac{1}{M}\, \delta\left(\frac{x_i}{M} + x_o, \frac{y_i}{M} + y_o\right)$$

$$\Rightarrow U_i(x_i, y_i) = \frac{1}{M}\, \tilde{U}_o\left(-\frac{x_i}{M}, -\frac{y_i}{M}\right)$$

MAKING VARIABLE SUBSTITUTION

$\tilde{x}_o \triangleq -M x_o$; $\tilde{y}_o \triangleq -M y_o$

$$\Rightarrow h(x_i, y_i, x_o, y_o) = M \int_{-\infty}^{\infty}\int P(\lambda d \hat{x}, \lambda d \hat{y})$$

$$\times e^{-j2\pi[(x_i - \tilde{x}_o)\hat{x} + (y_i - \tilde{y}_o)\hat{y}]} d\hat{x}\, d\hat{y}$$

$$\hat{\tilde{h}} = \frac{1}{M}\tilde{h}$$

$$U_i(x_i, y_i) = \iint_{-\infty}^{\infty} \hat{\tilde{h}}(x_i - \tilde{x}_o, y_i - \tilde{y}_o)\left(\frac{1}{M} U_o\left(\frac{\tilde{x}_o}{M}, \frac{\tilde{y}_o}{M}\right)\right) d\tilde{x}_o\, d\tilde{y}_o$$

= DIFFRACTION LIMITED IMAGE

$$= h(x_i, y_i) * U_g(x_i, y_i)$$

$$U_g(x_i, y_i) = \frac{1}{M} U_o\left(\frac{x_i}{M}, \frac{y_i}{M}\right)$$

10-16-75 (THURS.)

TOPICS CHOSEN FOR PRESENTATIONS

BROCK - THE IPPS IMAGING SYSTEM AT KITT PEAK OBSERV.

NACOL - ACOUSTICAL IMAGING FOR BIOMEDICAL APPLI

CHIN - INTEGRATED OPTICS

TUNG - WALSH FUNCTIONS AND THEIR APPL.

GROSS - BIOMEDICAL PATTERN RECOGNITION

JOSEPH - APPLIC. OF ODP TO ANALYSIS OF ERTS DAT

QUATRA - ACOUSTICAL HOLOGRAPHY

GRADING: SPOT GRADING

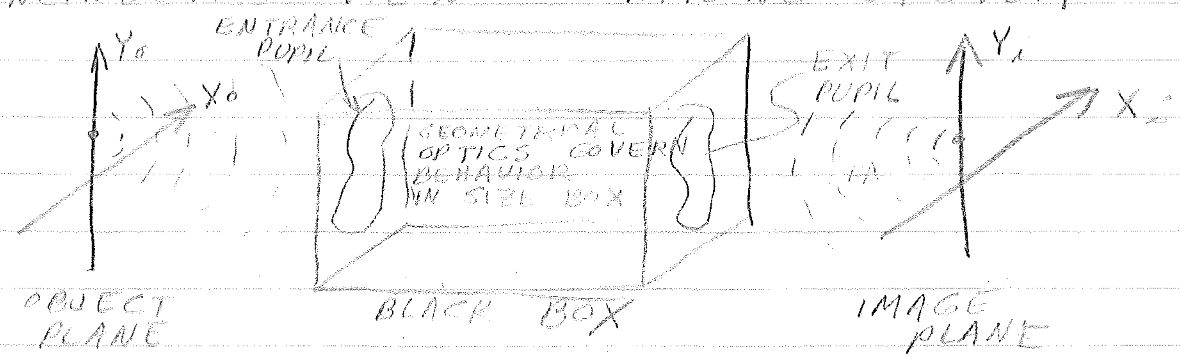
HOME WORK #5 DUE TUES, NOV. 4

CHAPT 6: 1, 2, 3, 5, 7, 8, 9

CHAPT. 6 : FREQ. ANALYSIS OF OPTICAL IMAGING SYSTEM

SPATIALLY INCOHERENT ILLUM
SPATIALLY COHERENT ILLUM } TWO EXTREMES
PARTIALLY COHERENT ILLUM ← IN BETWEEN
→ (BERAN & PARRENT - THEORY OF PARTIAL COHERENCE, PRENTICE HALL, 1964)

ENGINEER'S VIEW OF IMAGING SYSTEM



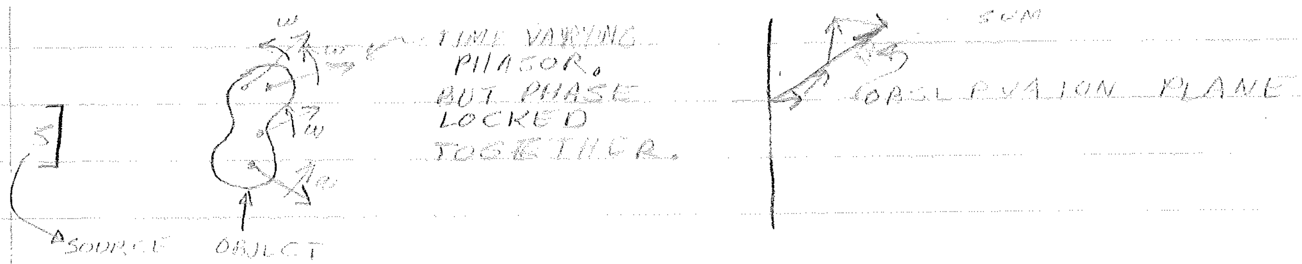
DIFFRACTION LIMITED: AN IMAGING SYSTEM IS SAID TO BE DIFFRACTION LIMITED IF A DIVERGING SPHERICAL WAVE, FROM ANY POINT SOURCE OBJECT, IS CONVERTED BY THE SYSTEM INTO A NEW WAVE, AGAIN PERFECTLY SPHERICAL, THAT CONVERGES TOWARD AN IDEAL POINT IN THE IMAGE PLANE. (WHEN YOU GO TO THAT IMAGE POINT TO WHERE THE WAVEFRONTS APPEAR TO CONVERGE, ONE SEES A DIFFRACTION PATTERN ASSOCIATED WITH THE EXIT (OR EQUIVALENTLY ENTRANCE) PUPIL)

ALL DIFFRACTION ASSOCIATED WITH PUPILS

ENTRANCE PUPIL - DEVELOPED BY ABBE

EXIT PUPIL - " " LORD RAYLEIGH

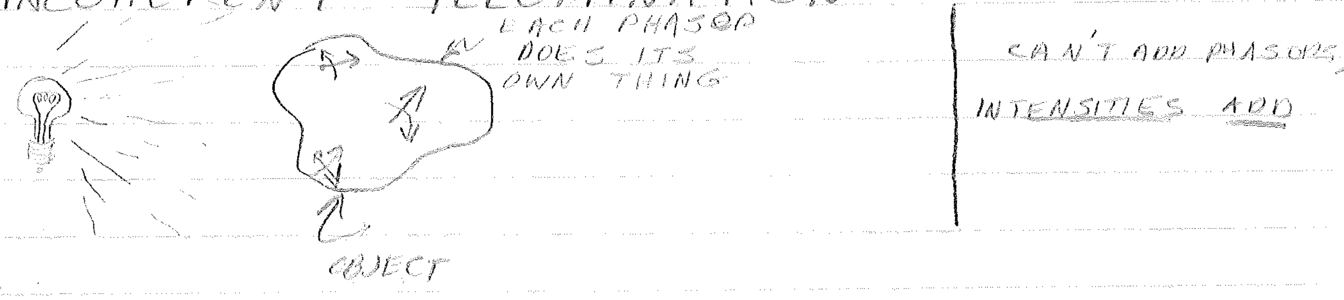
- COHERENT ILLUMINATION



PHASORS ROTATE ALL AT ω i.e. PHASE LOCKED.

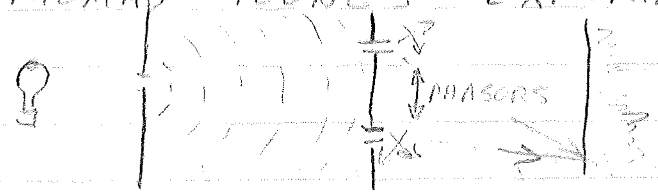
$\Rightarrow$ RELATIVE PHASES DON'T CHANGE WITH TIME.

- INCOHERENT ILLUMINATION



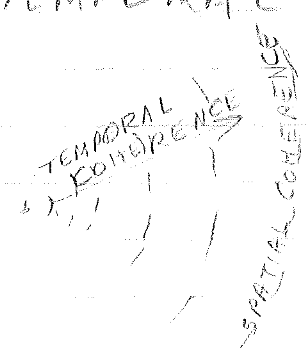
- TEMPORAL COHERENCE

THOMAS YOUNG'S EXPERIMENT



SPATIAL COHERENCE $\rightarrow$ GET FRINGES

NUMBER OF FRINGES FROM TEMPORAL COHERENCE.



LIGHT IS QUASI-MONOCHROMATIC

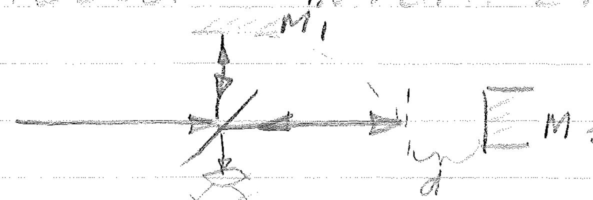
IF $\Delta\nu/\nu \ll 1$ (NARROWBAND SOURCE)

COHERENCE TIME $\approx \frac{1}{\Delta\nu}$

FOR INTERFERENCE, $\frac{1}{\Delta\nu} \gg \frac{d}{c}$

$\left\{\begin{array}{l} d = \text{MAXIMUM PATH LENGTH DIFFERENCE} \\ c = \text{SPEED OF LIGHT} \end{array}\right.$

THIS IS THE PRINCIPLE OF THE MICHELSON INTERFEROMETER



MAY USE TO MEASURE COHERENCE LENGTH = $c/\Delta\nu$ = C × COHERENCE TIME

10-28-75 (TUES)

SPATIALLY INCOHERENT - LINEAR IN INTENSITY

SPATIALLY COHERENT - LINEAR IN FIELD AMPLITUDE

FOR COHERENT

$$U_i = U_g * \tilde{h}$$

$$\tilde{h} = \mathcal{F}[P(\lambda d_i x, \lambda d_i y)]$$

$$I = |U_i|^2 = |U_g * h|^2$$

FOR INCOHERENT

$\langle U_g(\tilde{x}_0, \tilde{y}_0, t)\, U_g^*(\tilde{\tilde{x}}_0, \tilde{\tilde{y}}_0, t)\rangle$ = ∞ TIME AVERAGE

$$= \tilde{K} I_g(x_0, y_0)\, \delta(\tilde{x} - \tilde{\tilde{x}}_0, \tilde{y}_0 - \tilde{\tilde{y}}_0)$$

$$I_i = I_g * |h|^2$$

COHERENT SYSTEM (IMAGING)

$U_i(x,y) = U_g(x,y) * \tilde{h}(x,y)$

h (NORMALIZED) $= \tilde{h} = \int\int_{-\infty}^{\infty} P(\lambda \tilde{x} d_i, \lambda \tilde{y} d_i) e^{j2\pi(x\tilde{x}+y\tilde{y})} \, d\tilde{x} \, d\tilde{y}$

$G_i = \mathcal{F}(U_i)$

$\mathcal{F}[U_i] = \mathcal{F}[U_g] \, \mathcal{F}[h]$

$G_g = \mathcal{F}[U_g]$

$H = \mathcal{F}[h]$ = COHERENT TRANSFER FUNCTION

$= \mathcal{F}\,\mathcal{F}[P(\lambda x d_i, \lambda y d_i)]$

$= P(-\lambda f_x d_i, -\lambda f_y d_i)$

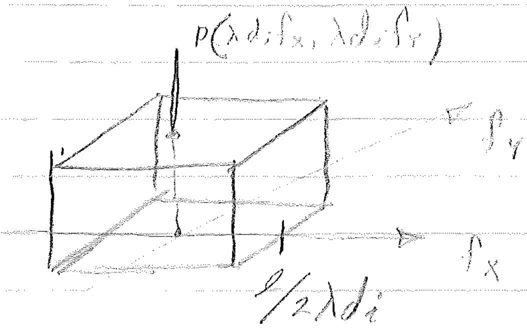
USUALLY DROP COORDININATE INVERSION

$\hat{H}(f_x, f_y) = P(\lambda d_i f_x, \lambda d_i f_i) \Longleftarrow$ CTF

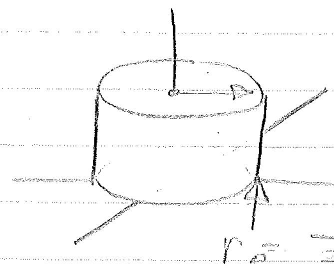
EX: LXL SQUARE PUPIL:

$$P(\lambda d_i f_x, \lambda d_i f_y) = \begin{cases} 1 & ; \begin{cases} \lambda d_i |f_x| \le l/2 \\ \lambda d_i |f_y| \le l/2 \end{cases} \\ 0 & ; \text{OTHERWISE} \end{cases}$$

$$= \begin{cases} 1 & ; \begin{cases} |f_x| \le l/2\lambda d_i \\ |f_y| \le l/2\lambda d_i \end{cases} \\ 0 & ; \text{OTHERWISE} \end{cases}$$



EX: CIRCULAR PUPIL



$r_0 = \frac{\ell}{2\lambda d_i}$ = COHERENT CUTOFF FREQUENCY

IN SUMMARY:

SQUARE: $H(f_X, f_Y) = \text{rect}\left(\frac{\lambda d_i f_X}{\ell}, \frac{\lambda d_i f_Y}{\ell}\right)$

CIRCLE: $H(f_X, f_Y) = \text{circ}\left(\frac{\sqrt{f_X^2 + f_Y^2}}{\ell / 2\lambda d_i}\right)$

INCOHERENT IMAGING

$$I_i(x, y) = I_g(x, y) \underset{2}{*} |\tilde{h}|^2$$

$$\mathcal{H} = \frac{\mathcal{F}[|h|^2]}{E} = \frac{\iint |\tilde{h}|^2 e^{-j2\pi(f_X x_i + f_Y y_i)} dx_i\, dy_i}{\iint_{-\infty}^{\infty} |h(x, y)|^2 dx_i\, dy_i}$$

NOTE:

$\mathcal{H}(0, 0) = 1$ $\mathcal{H}$ = OTF = OPTICAL TRANSFER FUNCTION

$$\mathcal{G}_i(f_X, f_Y) = \mathcal{F}\{I_i\} \Big/ \mathcal{F}\{I_i\}\big|_{f_X = 0 = f_Y}$$

$$\mathcal{G}_g(f_X, f_Y) = \mathcal{F}\{I_g\} \Big/ \mathcal{F}\{I_g\}\big|_{f_X = 0 = f_Y}$$

THUS: $\mathcal{G}_i = \mathcal{G}_g \mathcal{H}$

$|\mathcal{H}|$ = MODULATION TRANSFER FUNCTION (MTF)

$$\mathcal{H} = \frac{\mathcal{F}[|h|^2]}{\mathcal{F}[|h|^2]\big|_{f_X = f_Y = 0}} = \frac{\mathcal{F}[h] \star \mathcal{F}[h]}{\left[\mathcal{F}[h] \star \mathcal{F}[h]\right]\big|_{f_X = f_Y = 0}}$$

$$\mathcal{F}[h] = H \Rightarrow \text{CTF}$$
$$= P(\lambda d_i f_x, \lambda d_i f_x)$$

$$\mathcal{H} = \frac{H \star H}{(H \star H)|_{f_x = f_y = 0}}$$

PROPERTIES OF $\mathcal{H}$:

① $\mathcal{H}(0,0) = 1$

② $\mathcal{H}(-f_x, -f_y) = \mathcal{H}^*(f_x, f_y) \Leftarrow$ HERMETIAN

③ $|\mathcal{H}(f_x, f_y)| \le |\mathcal{H}(0,0)|$

FOR A SQUARE PUPIL:

$$\mathcal{H}(f_x, f_x) = \Lambda\left(\frac{f_x}{2f_0}\right) \Lambda\left(\frac{f_x}{2f_0}\right)$$

NOTE TWICE THE CUT-OFF.

10-30-75 (THURS)

$$\mathcal{H}(f_x) = \frac{\int_{-\infty}^{\infty} P\left(\xi - \frac{\lambda d_i f_x}{2}\right) P\left(\xi + \frac{\lambda d_i f_x}{2}\right) d\xi}{\int_{-\infty}^{\infty} P(\xi)\, d\xi}$$

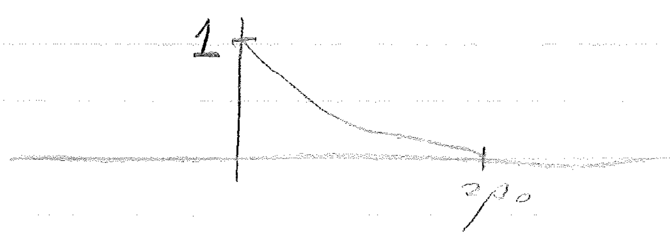
= OVERLAP AREA / TOTAL AREA

FOR SQARE PUPIL:

$$\mathcal{H}(f_x, f_y) = \Lambda\left(\frac{f_x}{2f_0}, \frac{f_y}{2f_0}\right)$$

FOR CIRCULAR PUPIL

$$\mathcal{H}(\rho) = \begin{cases} \frac{2}{\pi}\left[\cos^{-1}\left(\frac{\rho}{2\rho_0}\right) - \frac{\rho}{2\rho_0}\sqrt{1 - \left(\frac{\rho}{2\rho_0}\right)^2}\right] & \rho < 2\rho_0 \\ 0 & \end{cases}$$



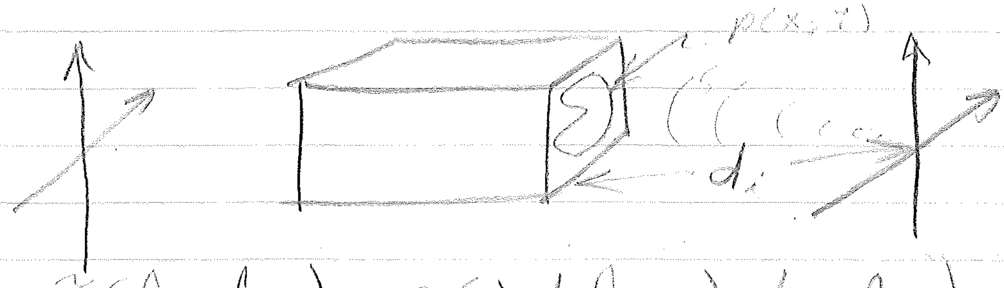
$\rho_0 = f_0$

NOTE: SAME CUTOFF

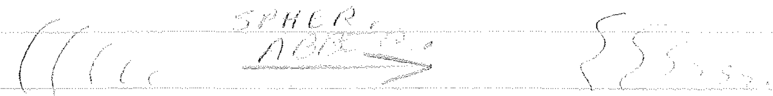
ABERRATIONS

DEFOCUS — SPHERICAL ABERRATION

COMA — TURBULENCE

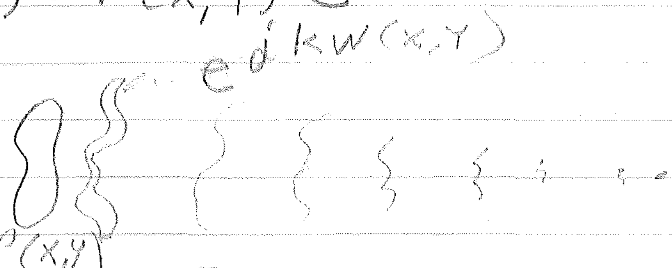


$$\tilde{H}(f_X, f_Y) = P(\lambda d_i f_X, \lambda d_i f_Y)$$



USE GENERALIZED PUPIL FUNCTION

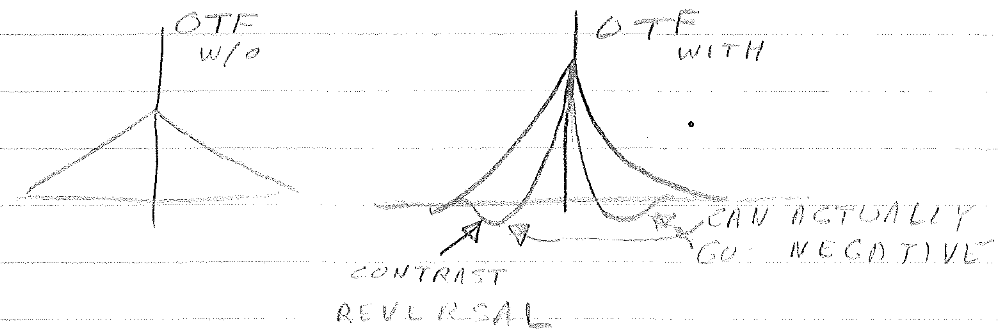
$$\tilde{P}(x,y) = P(x,y)\, e^{jkW(x,y)}$$



NEW CTF:

$$H(f_X, f_Y) = P(\lambda d_i f_X, \lambda d_i f_Y) \times e^{jkW(\lambda d_i f_X, \lambda d_i f_Y)}$$

FOR OTF, AUTOCORRELATE CTF



CONSIDER

$$I_i = I_g * |h^2|$$

$$\mathcal{F}[I_g] = \mathcal{F}[I_i] / \mathcal{F}[h^2]$$

$$\frac{1}{\mathcal{F}[|h^2|]} = \text{INVERSE FILTER}$$

COMPARISON OF COHERENT AND INCOH. SYSTEMS
(LOOK AT INTENSITY OF BOTH)

COHERENT: $U_i = U_g * h \Rightarrow I_i = |U_i|^2 = |U_g * h|^2$

$$\Rightarrow \mathcal{F}[I_i] = \mathcal{F}[|U_g * h|^2]$$

$$= \mathcal{F}[U_g * h] \star \mathcal{F}[U_g * h]$$

LET $\mathcal{F}\{U_g\} = G_g$

$\mathcal{F}[h\} = H \Leftarrow$ CTF

$$\Rightarrow \mathcal{F}[I_i] = (GH) \star (GH)$$

INCOHERENT: $I_i = I_g * |\hat{h}|^2$

$$\Rightarrow \mathcal{F}[I_i] = \mathcal{F}[I_g] \; \mathcal{F}[|\hat{h}|^2]$$

$$\mathcal{F}[I_g] = \mathcal{F}(|U_g|^2) = G_g \star G_g$$

$$\Rightarrow \mathcal{F}[I_i] = (G_g \star G_g)(H \star H)$$

INCOH: $\mathcal{F}[I] = (G_g \star G_g)(H \star H)$

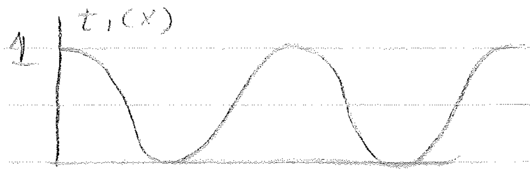
COHER: $\mathcal{F}[I] \quad GH \star GH$

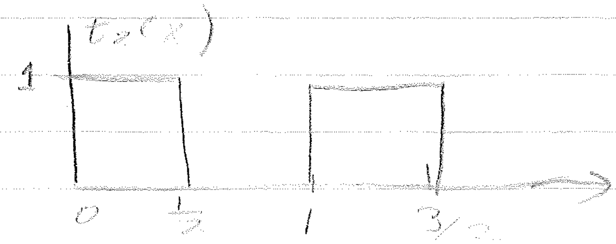
9-4-75 (MON)

HOMEWORK (CHAPT. 7: 1, 2, 5, 9, 11, 13) DUE TUES NOV 18

QUIZ:

PROBLEM 2





$t_1(x) = \frac{1}{2}(1 + \cos 2\pi x)$

(a) % ABSORBED

(b) % IN 0 ORDER

(c) % IN A SINGLE 1^{ST} ORDER COMPON.

a. % ABSORBED $= 100\left(1 - \int_0^1 |t(x)|^2 dx\right)$

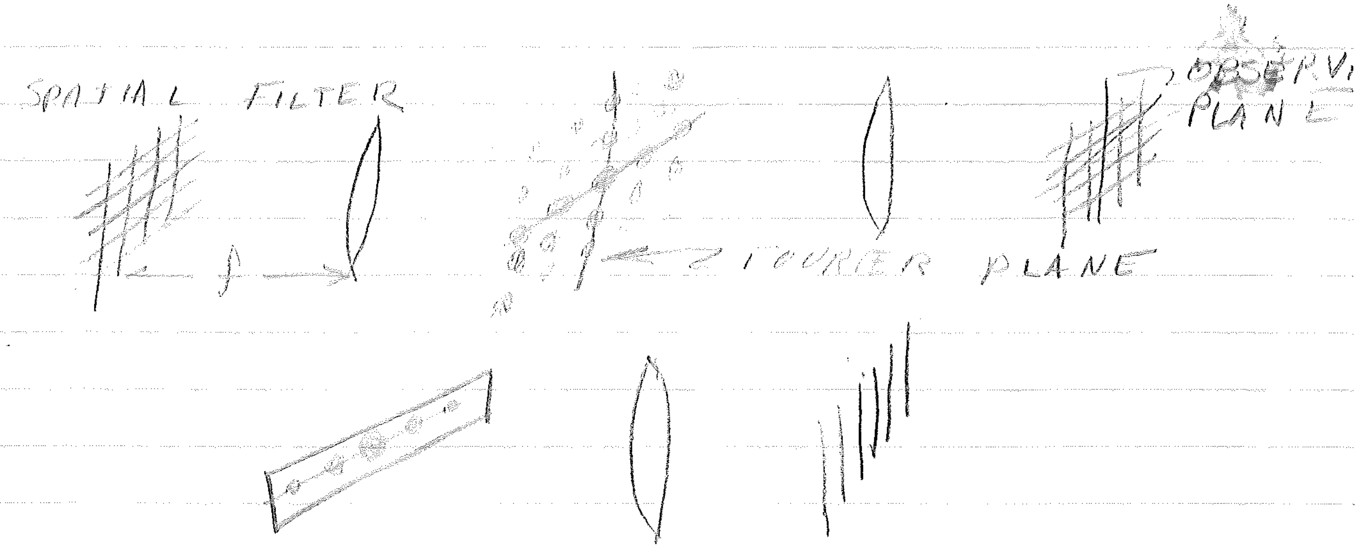
FOR $t_1(x)$, $\% = 100\left(1 - 3/8\right)$

FOR $t_2(x)$, $\% = 100\left(1 - \frac{1}{2}\right)$

b. 25% FOR EACH GRATING

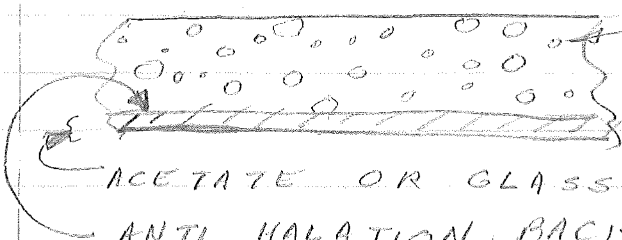
c. 6.25% FOR sin

10.1% FOR RECT



10-6-75 (THURS)

(PHOTOGRAPHIC FILM



SILVER HALIDE GRAINS (SUSPENDED IN AN EMULSION)

ACETATE OR GLASS BASE

ANTI HALATION BACKING (NOT REFLECTING)

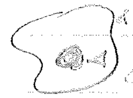
STEPS:

1. EXPOSURE = $E(x,y) = I(x,y)\,T$

T = EXPOSURE TIME

$I(x,y)$ = INCIDENT INTENSITY

2. LATENT IMAGE (SOME GRAINS ARE NOW DEVELOPABLE)



1: AgBr HAS CAPABILITY TO BE DEVELOPED

2: Ag

(EACH GRAIN HAS "TRESHOLD", SAY 4 PHOTONS.)

3. DEVELOPER - ALL DEVELOPABLE GRAINS ARE REDUCED TO Ag GRAINS)

4. FIX - CONVERTS UNDEVELOPED GRAINS INTO WATER SOLUABLE SALT

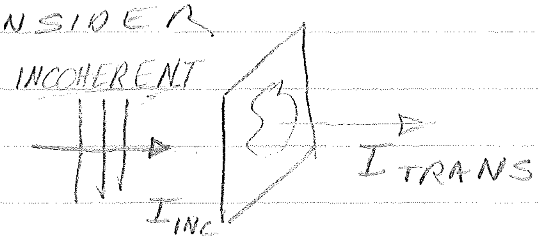
5. WASH

6. DRY

OTHER FACTORS ("ART")

EXPOSURE, DEVELOPER, DEVELOPMEN TIME, TEMP.

CONSIDER



$$\tau(x,y) \stackrel{\Delta}{=} \text{INTENSITY TRANSMITTANCE}$$

$$= \underset{\text{OF}}{\underset{\text{AVERAGE}}{\text{LOCAL}}} \left\{ \frac{I\,(\text{TRANSMITTED @ } (x,y))}{I\,(\text{INCIDENT @ } (x,y))} \right\}$$

AVERAGE OVER A AREA >> GRAIN AREA BUT << SCALE OF SPATIAL VARIATIONS OF INFO ON THE FILM.

PHOTOGRAPHIC DENSITY

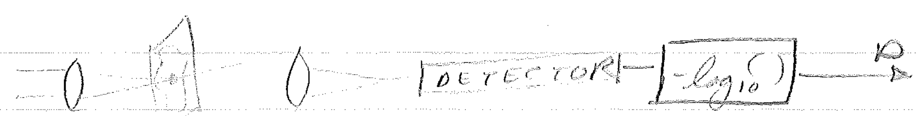
$$D(x,y) = \log_{10} \frac{1}{\tau(x,y)}$$

$$= -\log_{10} \tau(x,y)$$

$$\tau = 10^{-D}$$

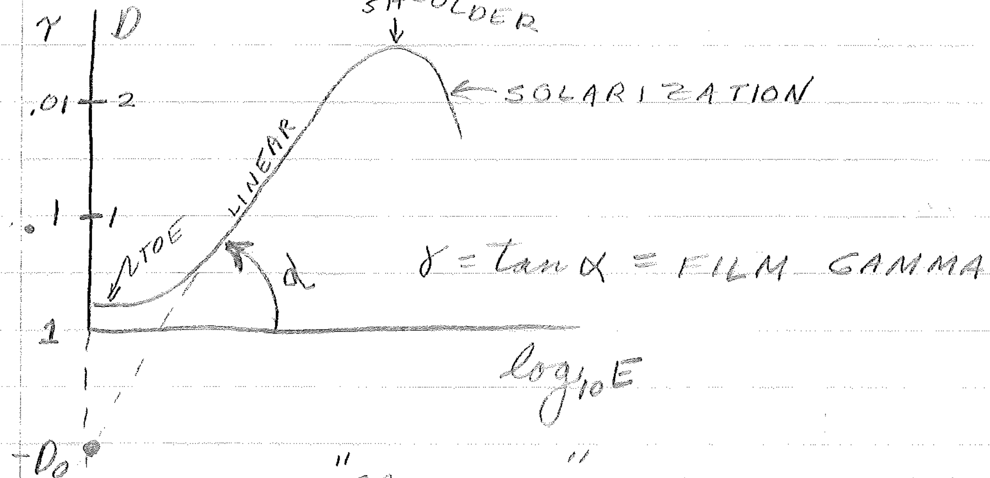
IN PRACTICE $0 < D < 5$

SCANNING:



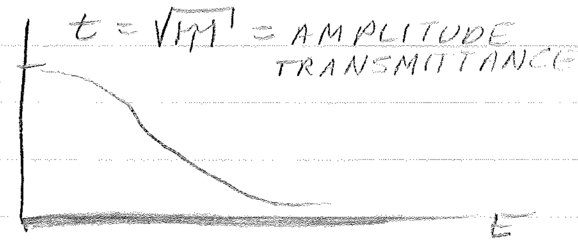
SCANNING MICRODENSITOMETERS

H & D CURVE (HURTER-DRIFFIELD)



TOE - "GROSS FOG" FROM THERMAL EXPOSURE

H & D CURVE CHANGES WITH DEVELOPMENT.



FINE GRAINS ⇒ LOW SENSITIVITY ⇒ LONG EXPOSURE (LoSpd.
LARGE GRAINS ⇒ HIGH SENSITIVITY ⇒ SHORT EXPOSURE (HiSpd

USING FILM IN AN INCOHERENT:

$$D = \gamma_n \log_{10} E - D_0 \Leftarrow \text{FROM H-D CURVE}$$

$$E = \mathcal{I} T$$

RECALL THAT $D = \log_{10}\frac{1}{\tau_n} = -\log_{10}\tau_n$

$$\Rightarrow \log_{10}\tau_n = -\gamma_n \log_{10}(\mathcal{I}T) + D_0$$

$$\tau_n = 10^{-\gamma_n \log_{10}(\mathcal{I}T) + D_0}$$
$$= 10^{D_0}(\mathcal{I}T)^{-\gamma_n}$$

$$K_n = 10^{D_0} T^{-\gamma_n}$$

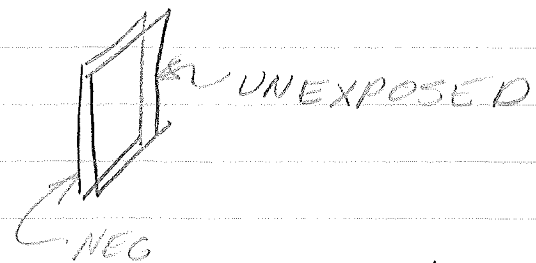
$$\Rightarrow \tau_n = K_n \mathcal{I}^{-\gamma_n} \Leftarrow$$ NON-LINEAR (A BUMMER)

TO GET γ LINEARLY RELATED TO $\mathscr{l}$

TWO STEP PROCESS

(1) MAKE A NEGATIVE

(2) EXPOSE NEW PIECE OF FILM THROUGH THE NEGATIVE & THUS CREATE A POSITIVE



$$\gamma_p = K_{n2} \left(I_0 \gamma_{n1} \right)^{-\delta_{n2}}$$

$$= K_{n2} \left(I_0^{-\delta_{n2}} K_{n1}^{-\delta_{n2}} \mathscr{l}^{\overbrace{\delta_{n1}\delta_{n2}}^{\delta_p}} \right)$$

MAY NOW MAKE $\delta_p = 1. \Rightarrow \gamma_p \sim \mathscr{l}$

J.F. Walkup

September ~~16~~ 11, ~~1974~~ 1975

EE 5360 (Fourier Optics and Holography)

Details on Project: Research/Oral Presentation on a Topic of Your Choice (Optics)

Listed below are some sample projects, but others may be chosen also. Feel free to consult with me on your choice of topic. If your MS or Ph.D. thesis work involves Fourier Optics or Holography, you may want to choose a topic relating to your thesis work. The key dates are:

Deadline for choice of topic: ~~Monday~~ Tues., ~~September 30~~ Oct. 14

Oral Presentations: Week of ~~November 18~~ Dec. 1

Potential topics: (Basic concepts and state of the art in some area):

1. Holographic interferometry and its applications
2. Color holography and its applications
3. Synthetic aperture imaging
4. Digital holography
5. Holographic memories for computers (read only, read-write)
6. New materials for holography
7. Restoration of turbulence-degraded images.
8. Acoustical imaging for biomedical applications
9. Integrated optics.
10. Optical communications: special problems and potentials
11. Modulatable lasers and their applications in optical communications
12. Walsh functions and their applications in optics (Walsh transforms, etc.)
13. Picture coding for bandwidth compression: the problem and some recent developments.
14. Pattern recognition applications with X-rays: applications in medicine ("black lung disease", "brown lung disease" (byssanosis).
15. Applications of optical data processing to the analysis of ERTS data (Earth resources technology sattelites)
16. Opportunities for EE's in the optics industry.
17. Space-variant image restoration techniques

Notes on the oral presentations: The talks should be about 30 minutes long plus 5 minutes for questions. They should be illustrated with whatever visual aids appear most appropriate (overhead projector, slides, opaque projector, etc.). Treat this as a presentation to a group familiar with the general field of Fourier optics and holography, but generally unfamiliar with the details of your topic. Make the talk somewhat tutorial-i.e.-don't snow us with lots of equations but use equations if the material calls for it. Don't talk in a monotone. We may have the rest of your fellow students rate you on your talk. More details on talk formats will be given later.

B.C. Brock
8-24-74

Proof of Goodman's Identity:

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ja\cos(\theta-\varphi)}\, d\theta$$

Use Schäfli's integral representation of Bessel function (Mathews & Walker, ch 7) the contour is once around origin in the positive sense. This is a result of the generating function.

Schäfli's integral:

$$J_n(a) = \frac{1}{2\pi i}\oint \frac{e^{\frac{a}{2}\left(t-\frac{1}{t}\right)}}{t^{n+1}}\, dt$$

Let: $t = -ie^{i(\theta-\varphi)}$ $\quad dt = e^{i(\theta-\varphi)}\, d\theta$

$$J_n(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{a/2\left[-ie^{i(\theta-\varphi)} - ie^{-i(\theta-\varphi)}\right]}\, e^{i(\theta-\varphi)}}{(-i)^{n+1}\, e^{i(\theta-\varphi)(n+1)}}\, d\theta$$

$$J_n(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ai\cos(\theta-\varphi) - ni(\theta-\varphi)}}{(-1)^{n+1}(i)^{n+2}}\, d\theta$$

In particular:

$$J_0(a) = \frac{1}{2\pi}\int_0^{\pi} e^{-ia\cos(\theta-\varphi)}\, d\theta$$

Solutions

2-3a)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^2 \tilde{g}(x,y) = \nabla^2 \iint_{-\infty}^{\infty} \tilde{G}(f_x, f_y)\, e^{j2\pi(f_x x + f_y y)}\, df_x\, df_y$$

$$= \iint_{-\infty}^{\infty} \tilde{G}(f_x, f_y)\, \nabla^2 \left[e^{j2\pi(f_x x + f_y y)} \right] df_x\, df_y$$

$$= \iint_{-\infty}^{\infty} \left[(j2\pi f_x)^2 + (j2\pi f_y)^2 \right] \tilde{G}(f_x, f_y)\, e^{j2\pi(f_x x + f_y y)}\, df_x\, df_y$$

$$\therefore\ \nabla^2 \tilde{g}(x,y) = \mathcal{F}^{-1}\left\{ -4\pi^2 (f_x^2 + f_y^2)\, \tilde{G}(f_x, f_y) \right\}$$

$$\Rightarrow\ \mathcal{F}\left\{ \nabla^2 \tilde{g}(x,y) \right\} = -4\pi^2 (f_x^2 + f_y^2)\, \tilde{G}(f_x, f_y)$$

$$\boxed{\mathcal{F}\left\{ \nabla^2 \tilde{g}(x,y) \right\} = -4\pi^2 (f_x^2 + f_y^2)\, \mathcal{F}\left\{ \tilde{g}(x,y) \right\}}$$

2-4

(a) $\tilde{g}_R(r) = \delta(r - r_0)$

$$\mathcal{B}\{\delta(r-r_0)\} = 2\pi \int_0^\infty r\,\delta(r-r_0)\, J_0(2\pi r\rho)\,dr$$

$$\boxed{\mathcal{B}\{\delta(r-r_0)\} = 2\pi r_0 J_0(2\pi r_0 \rho)}$$

By sifting property of the delta function

(b) $\tilde{g}_R(r) = \begin{cases} 1, & a \le r \le 1 \\ 0, & \text{elsewhere} \end{cases}$

or $\tilde{g}_R(r) = \operatorname{circ}(r) - \operatorname{circ}\left(\frac{r}{a}\right)$

$$\mathcal{B}\{\tilde{g}_R(r)\} = \frac{J_1(2\pi\rho)}{\rho} - \frac{a^2 J_1(2\pi a\rho)}{a\rho}$$

(using Relation on p 11 of text)

$$\boxed{\mathcal{B}\{\tilde{g}_R(r)\} = \frac{J_1(2\pi\rho) - a\,J_1(2\pi a\rho)}{\rho}}$$

(c) $$2\pi \int_0^\infty r\,\tilde{g}_R(ar)\, J_0(2\pi r\rho)\,dr = 2\pi \int_0^\infty \frac{r'}{a}\, g_R(r')\, J_0\left(\frac{2\pi r'\rho}{a}\right)\frac{dr'}{a}$$

where $r' = ar$ $r = r'/a$

Thus $$\boxed{\mathcal{B}\{\tilde{g}_R(ar)\} = \frac{1}{a^2}\, \tilde{G}\left(\frac{\rho}{a}\right)}$$

(d) $e^{-\pi r^2} = e^{-\pi(x^2+y^2)}$ The Fourier-Bessel Transform is just a special case of the F.T.

So $\mathcal{B}\{e^{-\pi r^2}\} = \mathcal{F}\{e^{-\pi(x^2+y^2)}\} = e^{-\pi(f_X^2 + f_Y^2)}$

for circularly symmetric functions

$$\boxed{\mathcal{B}\{e^{-\pi r^2}\} = e^{-\pi\rho^2}}$$

2-5

$$\tilde{p}(x,y) = \tilde{g}(x,y) ** \left[\text{comb}\left(\frac{x}{X}\right)\text{comb}\left(\frac{y}{Y}\right)\right]$$

(a) Fourier transform using the convolution and <u>similarity</u> theorems (pp 9-10)

$$\tilde{P}(f_x, f_y) = \tilde{G}(f_x, f_y)\, X Y \,\text{comb}\, X f_x \,\text{comb}\, Y f_y$$

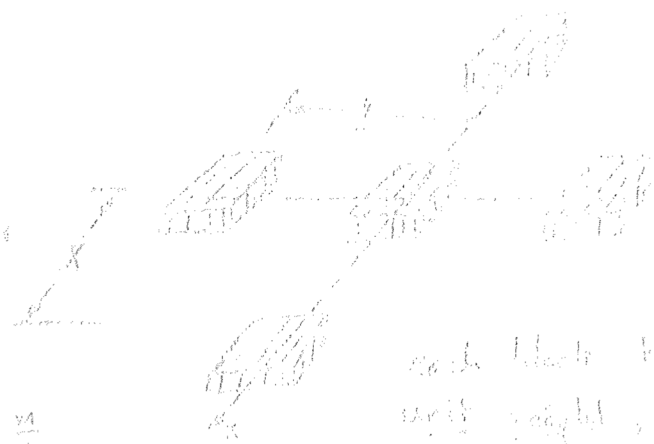
using the result of Prob 2-2 (?)

$$X Y \,\text{comb}\, X f_x \,\text{comb}\, Y f_y = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

$$\Rightarrow \quad \boxed{\tilde{P}(f_x, f_y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \tilde{G}\left(\frac{n}{X}, \frac{m}{Y}\right)\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)}$$

(b) $\tilde{g}(x,y) = \text{Rect}\left(2\frac{x}{X}\right)\text{Rect}\left(2\frac{y}{Y}\right)$

$\tilde{p}(x,y)$ is a replicated version of $\tilde{g}(x,y)$ i.e. we take $\tilde{g}(x,y)$ and center it at $(x = nX, y = mY)$, for $n = 0, \pm 1, \pm 2, \ldots$ $m = 0, \pm 1, \pm 2, \ldots$ Then the blocks are all over the x-y plane.



$$\tilde{G}(f_x, f_y) = \frac{X}{2}\frac{Y}{2}\,\text{sinc}\,\frac{X}{2}f_x\,\text{sinc}\,\frac{Y}{2}f_y$$

each block has unit height, [illegible]

$$\therefore \quad \tilde{G}\left(\frac{n}{X}, \frac{m}{Y}\right) = \frac{XY}{4}\,\text{sinc}\,\frac{n}{2}\,\text{sinc}\,\frac{m}{2}$$

$$\boxed{\tilde{P}(f_x, f_y) = \frac{XY}{4}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\text{sinc}\,\frac{n}{2}\,\text{sinc}\,\frac{m}{2}\,\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)}$$

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2-6

(a) First, recall the F.T. relationships: eqs (2-1) & (2-2) from text

$$\tilde{G}(f_x, f_y) = \mathcal{F}\{\tilde{g}\} = \iint_{-\infty}^{\infty} \tilde{g}(x,y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\, dy \qquad (2\text{-}1)$$

$$\tilde{g}(x,y) = \mathcal{F}^{-1}\{\tilde{G}\} = \iint_{-\infty}^{\infty} \tilde{G}(f_x, f_y)\, e^{j2\pi(f_x x + f_y y)}\, df_x\, df_y \qquad (2\text{-}2)$$

We are given the relationships for $\mathcal{F}_A\{\tilde{g}\}$ and $\mathcal{F}_B\{\tilde{g}\}$ and we want an expression for $\mathcal{F}_B\{\mathcal{F}_A\{\tilde{g}\}\}$.

Write

$$\mathcal{F}_B\{\mathcal{F}_A\{\tilde{g}\}\} = \frac{1}{b}\iint_{-\infty}^{\infty}\left[\frac{1}{a}\iint_{-\infty}^{\infty} \tilde{g}(x,y)\, e^{-j2\pi\left[(\frac{f_x}{a})x + (\frac{f_y}{a})y\right]}\, dx\, dy\right] e^{j2\pi\left[(\frac{\alpha}{b})f_x + (\frac{\beta}{b})f_y\right]}\, df_x\, df_y$$

Now, using (2-1) above, we reduce the innermost integral in the preceeding equation and we get:

$$\mathcal{F}_B\{\mathcal{F}_A\{\tilde{g}\}\} = \frac{1}{b}\iint_{-\infty}^{\infty} \frac{1}{a}\,\tilde{G}\left(\frac{f_x}{a}, \frac{f_y}{a}\right) e^{j2\pi\left[(\frac{\alpha}{b})f_x + (\frac{\beta}{b})f_y\right]}\, df_x\, df_y$$

$$= \frac{a^2}{ab}\iint_{-\infty}^{\infty} \tilde{G}\left(\frac{f_x}{a}, \frac{f_y}{a}\right) e^{j2\pi\left[(\frac{a\alpha}{b})\frac{f_x}{a} + (\frac{a\beta}{b})\frac{f_y}{a}\right]}\, d\left(\frac{f_x}{a}\right) d\left(\frac{f_y}{a}\right)$$

Now, we use (2-2) and get

$$= \frac{a}{b}\,\tilde{g}\left(-\frac{a\alpha}{b}, -\frac{a\beta}{b}\right)$$

Thus $\boxed{\mathcal{F}_B\{\mathcal{F}_A\{\tilde{g}(x,y)\}\} = \frac{a}{b}\,\tilde{g}\left(-\frac{a}{b}x, -\frac{a}{b}y\right)}$

(b) For $a < b$, the transformation attenuates the optical signal, expands and reverses its space coordinates (magnification).

For $a > b$, the transformation amplifies the optical signal, shrinks and reverses the space coordinates (demagnification).

2-7 (a) $\tilde{g}(r,\theta) = \tilde{g}_R(r)\, e^{jm\theta}$

$$\tilde{G}(\rho,\phi) = \int_0^{2\pi} d\theta\, e^{jm\theta} \int_0^{\infty} dr\, r\, \tilde{g}_R(r)\, e^{-j2\pi r\rho\cos(\theta-\phi)}$$

$$= \int_0^{\infty} dr\, r\, \tilde{g}_R(r) \int_0^{2\pi} d\theta\, e^{jm\theta}\, e^{-j2\pi r\rho\sin(\theta-\phi+\frac{\pi}{2})}$$

Now using

$$\exp(-ja\sin x) = \sum_{k=-\infty}^{\infty} J_k(a)\, e^{-jkx}$$

to write

$$\tilde{G}(\rho,\phi) = \int_0^{\infty} dr\, r\, \tilde{g}_R(r) \int_0^{2\pi} \sum_{k=-\infty}^{\infty} J_k(2\pi r\rho)\, e^{-jk(\theta-\phi+\frac{\pi}{2})}\, e^{jm\theta}\, d\theta$$

$$= \int_0^{\infty} dr\, r\, \tilde{g}_R(r) \sum_{k=-\infty}^{\infty} J_k(2\pi r\rho)\, e^{jk\phi}\, e^{-jk\frac{\pi}{2}} \int_0^{2\pi} e^{-j(k-m)\theta}\, d\theta$$

However

$$\int_0^{2\pi} e^{-j(k-m)\theta}\, d\theta = \begin{cases} 2\pi, & k = m \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \quad \tilde{G}(\rho,\phi) = e^{jm\phi}\, e^{-j\frac{m\pi}{2}}\, 2\pi \int_0^{\infty} r\, \tilde{g}_R(r)\, J_m(2\pi r\rho)\, dr$$

Now $e^{-j\frac{m\pi}{2}} = (-j)^m$ for integer value of m.

$$\therefore \quad \boxed{\begin{aligned} \tilde{G}(\rho,\phi) &= (-j)^m e^{jm\phi} \left\{ 2\pi \int_0^{\infty} r\, \tilde{g}_R(r)\, J_m(2\pi r\rho)\, dr \right\} \\ &= (-j)^m e^{jm\phi}\, \mathcal{H}_m\{\tilde{g}_R(r)\} \end{aligned}}$$

where $\mathcal{H}_m\{\cdot\}$ is an m^{th} order Hankel Transform.

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2-7 cont'd

(b) $\tilde{g}(r,\theta) = \tilde{g}_R(r)\,\tilde{g}_\Theta(\theta)$

Now $\tilde{g}_\Theta(\theta)$ is periodic with period 2π

$\therefore$ $\tilde{g}_\Theta(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{jk\theta}$ where $c_k = \frac{1}{2\pi}\int_0^{2\pi} \tilde{g}_\Theta(\theta)\, e^{-jk\theta}\, d\theta$

(i.e. expandable in a complex Fourier Series)

Thus

$$\tilde{g}_R(r)\,\tilde{g}_\Theta(\theta) = \sum_{k=-\infty}^{\infty} c_k\, \tilde{g}_R(r)\, e^{jk\theta}$$

$$\Longrightarrow \mathcal{F}\{\tilde{g}_R(r)\,\tilde{g}_\Theta(\theta)\} = \sum_{k=-\infty}^{\infty} \mathcal{F}\{c_k\, \tilde{g}_R(r)\, e^{jk\theta}\}$$

using the Results of part (a) of 2-7 we obtain

$$\boxed{\tilde{G}_o(\rho,\phi) = \sum_{k=-\infty}^{\infty} (-j)^k c_k\, e^{jk\phi}\, \mathcal{H}_k\{\tilde{g}_R(r)\}}$$

where $\mathcal{H}_k\{\cdot\}$ is a k^{th} order Hankel Transform.

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2-8

If the system is invariant, then

$$e^{j2\pi(f_x x + f_y y)} \longrightarrow |\tilde{H}(f_x, f_y)|\, e^{j\phi(f_x, f_y)}\, e^{j2\pi(f_x x + f_y y)}$$

where $e^{j\phi(f_x, f_y)} = \angle \tilde{H}(f_x, f_y)$

We can write

$$\cos 2\pi(f_x x + f_y y) = \frac{1}{2}\left[e^{j2\pi(f_x x + f_y y)} + e^{-j2\pi(f_x x + f_y y)}\right]$$

Thus, if $\tilde{H}(-f_x, -f_y) = \tilde{H}^*(f_x, f_y)$ we get

$$\cos 2\pi(f_x x + f_y y) \longrightarrow |\tilde{H}(f_x, f_y)| \cos[2\pi(f_x x + f_y y) + \phi(f_x, f_y)]$$

So the sufficient conditions are:

1. Linear and invariant
2. $\tilde{H}(-f_x, -f_y) = \tilde{H}^*(f_x, f_y)$

   or $\tilde{h}(x, y)$ Real

3-1

$$\tilde{U}(r) = \frac{e^{jkr}}{r}$$ (see Fig. 3.4 for diag. of surface)

Consider the situation on the large sphere S_2, as radius R grows large.

$$\frac{\partial \tilde{U}}{\partial n} = \cos(n, r)\left\{jk - \frac{1}{R}\right\}\frac{e^{jkR}}{R}$$

$$\approx jk\,\frac{e^{jkR}}{R} - \frac{e^{jkR}}{R^2}$$

Thus

$$\frac{\partial \tilde{U}}{\partial n} - jk\tilde{U} = jk\,\frac{e^{jkR}}{R} - \frac{e^{jkR}}{R^2} - jk\,\frac{e^{jkR}}{R}$$

(1st & last terms cancel)

$$\Rightarrow \quad \frac{\partial \tilde{U}}{\partial n} - jk\tilde{U} = -\frac{e^{jkR}}{R^2}$$

Now $|e^{jkR}| \leq 1$ for all R

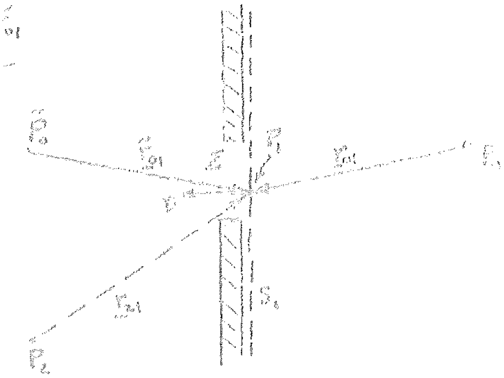
$$\Rightarrow \quad \lim_{R\to\infty} R\left(\frac{\partial \tilde{U}}{\partial n} - jk\tilde{U}\right) = \lim_{R\to\infty} \frac{e^{jkR}}{R} = 0$$

Thus

Sommerfeld Radiation condition is satisfied.

3-3

$$\tilde{G}_+(P_1) = \frac{e^{jkr_{01}}}{r_{01}} + \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}}$$



a)

$$\frac{\partial G_+}{\partial n} = \cos(\underline{n}, \underline{r}_{01}) \frac{e^{jkr_{01}}}{r_{01}} \left(jk - \frac{1}{r_{01}}\right) + \cos(\underline{n}, \tilde{\underline{r}}_{01}) \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}} \left(jk - \frac{1}{\tilde{r}_{01}}\right)$$

on S_1, $r_{01} = \tilde{r}_{01}$, $\cos(\underline{n}, \underline{r}_{01}) = -\cos(\underline{n}, \tilde{\underline{r}}_{01})$

$\therefore$ Substituting yeilds $\left.\frac{\partial G_+}{\partial n}\right|_{\text{on } S_1} = 0$

b. From Eq (3-15)

$$\tilde{U}(P_0) = \frac{1}{4\pi} \iint_{S_1} \left(\frac{\partial \tilde{U}}{\partial n} \tilde{G}_+ - \cancel{\tilde{U} \frac{\partial \tilde{G}_+}{\partial n}} \right) ds$$

$$\tilde{U}(P_0) = \frac{1}{4\pi} \iint_{S_1} \left[2 \frac{e^{jkr_{01}}}{r_{01}} \right] \frac{\partial \tilde{U}}{\partial n} \, ds$$

Assume

① That on S_1 $\frac{\partial \tilde{U}}{\partial n} = 0$ in the shadow

② That on S_1 $\frac{\partial \tilde{U}}{\partial n}$ has the same value across Σ as it would in the screen's absense.

Then

$$\boxed{\tilde{U}(P_0) = \frac{1}{2\pi} \iint_{\Sigma} \frac{\partial \tilde{U}(P_1)}{\partial n} \frac{e^{jkr_{01}}}{r_{01}} \, ds}$$

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3-3 cont'd

c.
$$\tilde{U}(P_1) = \frac{e^{jkr_{21}}}{r_{21}}$$

$$\frac{\partial \tilde{U}}{\partial n} = \cos(\underline{n}, \underline{r}_{21}) \left\{ jk - \frac{1}{r_{21}} \right\} \frac{e^{jkr_{21}}}{r_{21}}$$

Since $r_{21} \gg \lambda$

$$\frac{\partial \tilde{U}}{\partial n} \approx \cos(\underline{n}, \underline{r}_{21})\, jk \frac{e^{jkr_{21}}}{r_{21}}$$

Therefore

$$\boxed{\tilde{U}(P_0) = -\frac{1}{j\lambda} \iint_{\Sigma} \frac{e^{jk(r_{01}+r_{21})}}{r_{01}\, r_{21}} \cos(\underline{n}, \underline{r}_{21})\, ds}$$

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3-4

a. Circular Aperture

$$\tilde{t}(r) = \text{circ}\left(\frac{r}{d/2}\right)$$

$$\mathcal{F}\{\tilde{t}(r)\} = \tilde{T}(\rho) = \left(\frac{d}{2}\right)^2 \frac{J_1\left(2\pi \frac{d}{2} \rho\right)}{\frac{d}{2}\rho}$$

$$\boxed{\therefore\ \tilde{T}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \left(\frac{d}{2}\right)^2 \frac{J_1\left(2\pi \cdot \frac{d}{2} \frac{\sqrt{\alpha^2+\beta^2}}{\lambda}\right)}{\frac{d}{2} \frac{\sqrt{\alpha^2+\beta^2}}{\lambda}}}$$

b.

$$\tilde{t}(r) = 1 - \text{circ}\left(\frac{r}{d/2}\right)$$

$$\tilde{T}(\rho) = \delta(\rho) - \left(\frac{d}{2}\right)^2 \frac{J_1\left(2\pi \cdot \frac{d}{2} \rho\right)}{\frac{d}{2}\rho}$$

$$\boxed{\tilde{T}\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) = \delta\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) - \left(\frac{d}{2}\right)^2 \frac{J_1\left(2\pi \cdot \frac{d}{2} \frac{\sqrt{\alpha^2+\beta^2}}{\lambda}\right)}{\frac{d}{2} \frac{\sqrt{\alpha^2+\beta^2}}{\lambda}}}$$

3-5

We have

$$\tilde{u}_-(P,t) = \int_{-\infty}^{\infty} \tilde{U}(P,\nu)\, e^{j2\pi\nu t}\, d\nu$$

where $\tilde{U}(P,\nu) = \mathcal{F}\{\tilde{u}(P,t)\}$

making a change of variable

let $\nu' = -\nu$

$$\tilde{u}_-(P,t) = -\int_{\infty}^{0} \tilde{U}(P,-\nu')\, e^{-j2\pi\nu' t}\, d\nu'$$

$$= \int_{0}^{\infty} \tilde{U}(P,-\nu')\, e^{-j2\pi\nu' t}\, d\nu' \qquad (1)$$

This is a linear combination of an infinite number of monochromatic sources (waves), each with freq. ν' and complex amplitute $\tilde{U}(P,-\nu')$. Substituting eq (3-32) in text in (1) above yeilds

$$\tilde{u}_-(P_0,t) = \int_0^{\infty} \frac{1}{j\lambda}\left[\iint_{\Sigma} \tilde{U}(P_1,-\nu')\, \frac{e^{j\frac{2\pi\nu' r_{01}}{c}}}{r_{01}} \cos(\underline{n},\underline{r}_{01})\, ds\right] e^{-j2\pi\nu' t}\, d\nu'$$

interchanging the order of integration yeilds:

$$\tilde{U}_-(P_0,t) = \iint_{\Sigma} \frac{\cos(\underline{n},\underline{r}_{01})}{jc\, r_{01}} \left[\int_0^{\infty} \tilde{U}(P_1,-\nu')\,\nu'\, e^{j\frac{2\pi\nu' r_{01}}{c}}\, e^{-j2\pi\nu' t}\, d\nu'\right] ds$$

Now replace ν for $-\nu'$

$$= \iint_{\Sigma} \frac{\cos(\underline{n},\underline{r}_{01})}{jc\, r_{01}} \left[\int_{-\infty}^{0} \underbrace{\tilde{U}(P_1,\nu)(-\nu)\, e^{-j\frac{2\pi\nu r_{01}}{c}}}_{}\, e^{j2\pi\nu t}\, d\nu\right] ds$$

The quantity in [·] above is a form of the inverse F.T. of $\underbrace{\quad}$ for negitive ν. Since $\tilde{U}(P_1,\nu)$ is a non-monochrom wave which is nonzero only in a small region $\Delta\nu'$ around $\bar{\nu}$ $(\Delta\nu \ll \bar{\nu})$, we can say that ν is essentially a constant over the nonzero frequency spectrum around $-\bar{\nu}$

3-5 cont'd

Also if we let $\nu = -\bar{\nu}$ we can approximate

$$e^{-j\frac{2\pi\nu r_{01}}{c}} \approx e^{j\frac{2\pi\bar{\nu} r_{01}}{c}} \quad \text{for} \quad \frac{r_{01}\,\Delta\nu}{c} \ll 1 .$$

The percentage of error of this approx. is:

$$\frac{e^{j2\pi(\bar{\nu}+\Delta\nu)\frac{r_{01}}{c}} - e^{j2\pi\bar{\nu}\frac{r_{01}}{c}}}{e^{j2\pi\bar{\nu}\frac{r_{01}}{c}}} \approx e^{j\frac{2\pi\Delta\nu r_{01}}{c}} - 1$$

$$\approx 0$$

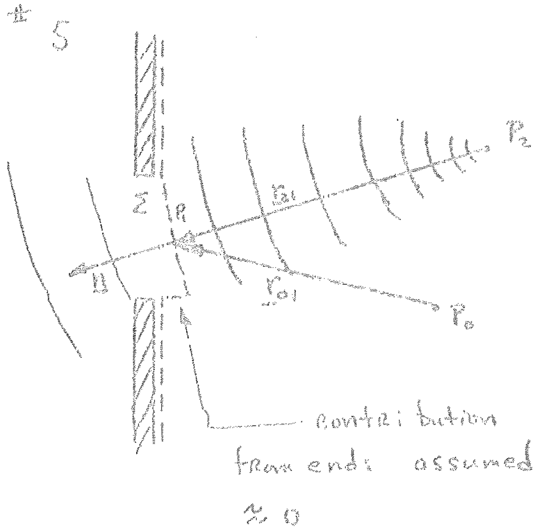
So we have.

$$\tilde{u}_-(P_0,t) = \iint_{\Sigma} \frac{\cos(\underline{n},\underline{r}_{01})}{jc\,r_{01}}(\bar{\nu})\, e^{j\frac{2\pi\bar{\nu} r_{01}}{c}} \Bigg[\underbrace{\int_{-\infty}^{0} \tilde{U}(P_1,\nu)\, e^{j2\pi\nu t}\, d\nu}_{\tilde{u}_-(P_1,t)}\Bigg] ds$$

and Remembering that $\frac{\bar{\nu}}{c} = \frac{1}{\bar{\lambda}}$ $\quad \bar{k} = \frac{2\pi\bar{\nu}}{c}$

$$\boxed{\tilde{u}_-(P_0,t) = \frac{1}{j\bar{\lambda}} \iint_{\Sigma} \tilde{u}_-(P_1,t)\, \frac{e^{j\bar{k} r_{01}}}{r_{01}} \cos(\underline{n},\underline{r}_{01})\, ds}$$

Problem # 5



Eq. 3-16

$$\tilde{U}(P_0) = \frac{1}{4\pi}\iint_{\Sigma}\left(\frac{\partial \tilde{U}}{\partial n}\tilde{G} - \tilde{U}\frac{\partial \tilde{G}}{\partial n}\right) ds$$

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P. # 5 contd

on Σ

$$\tilde{G}(P_1) = \frac{e^{jkr_{01}}}{r_{01}}$$

$$\frac{\partial \tilde{G}}{\partial n} = \cos(\underline{n}, \underline{r}_{01})\left[jk - \frac{1}{r_{01}}\right]\frac{e^{jkr_{01}}}{r_{01}}$$

$$\frac{\partial \tilde{G}}{\partial n} \approx jk\,\frac{e^{jkr_{01}}}{r_{01}}\cos(\underline{n}, \underline{r}_{01}) \qquad r_{01} \gg \lambda$$

Also

$$\tilde{U}(P_1) = A\,\frac{e^{-jkr_{21}}}{r_{21}}$$

where the minus is required to distinguish a converging spherical wave from a diverging spherical wave.

$$\frac{\partial \tilde{U}}{\partial n} = \cos(\underline{n}, \underline{r}_{21})\left[-jk - \frac{1}{r_{21}}\right] A\,\frac{e^{jkr_{21}}}{r_{21}}$$

but $\cos(\underline{n}, \underline{r}_{21}) = 1$

assuming $r_{21} \gg \lambda$

$$\frac{\partial \tilde{U}}{\partial n} = -jk\,A\,\frac{e^{jkr_{21}}}{r_{21}} = -jk\,\tilde{U}(P_1)$$

Thus

$$\tilde{U}(P_0) = \frac{1}{4\pi}\iint_{\Sigma}\left[-jk\,\tilde{U}(P_1)\frac{e^{jkr_{01}}}{r_{01}} - jk\,\tilde{U}(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos(\underline{n}, \underline{r}_{01})\right]ds$$

$$\boxed{\tilde{U}(P_0) = \frac{1}{j\lambda}\iint_{\Sigma}\tilde{U}(P_1)\,\frac{e^{jkr_{01}}}{r_{01}}\left[\frac{1 + \cos(\underline{n}, \underline{r}_{01})}{2}\right]ds}$$

Solutions

4-1

a. $t(x_1, y_1) = \text{Rect}\left(\frac{x_1}{X}\right) \text{Rect}\left(\frac{y_1}{Y}\right) * \left[\delta(x_1, y_1 - \frac{\Delta}{2}) + \delta(x_1, y_1 + \frac{\Delta}{2})\right]$

$$\therefore \mathcal{F}\{t(x_1, y_1)\} = 2XY \,\text{sinc}\, X f_x \,\text{sinc}\, Y f_y \left[\frac{e^{-j2\pi f_y \frac{\Delta}{2}} + e^{j2\pi f_y \frac{\Delta}{2}}}{2}\right]$$

$$\mathcal{F}\{t(x_1, y_1)\} = 2XY \,\text{sinc}\, X f_x \,\text{sinc}\, Y f_y \cos \pi f_y \Delta$$

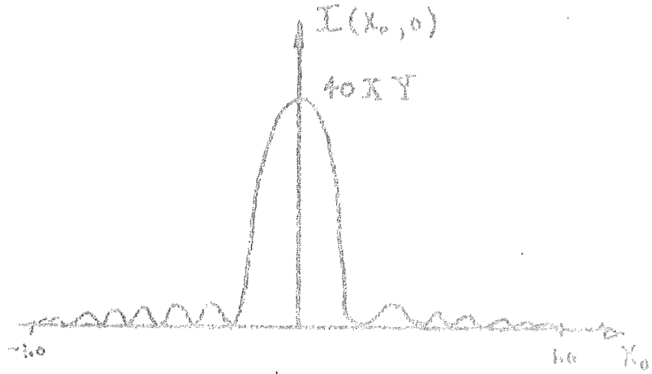
Recall the Intensity distribution is given by

$$I(x_0, y_0) = \frac{1}{\lambda^2 z^2} \left| \mathcal{F}\{t(x_1, y_1)\} \right|^2_{f_x = \frac{x_0}{\lambda z} \quad f_y = \frac{y_0}{\lambda z}}$$

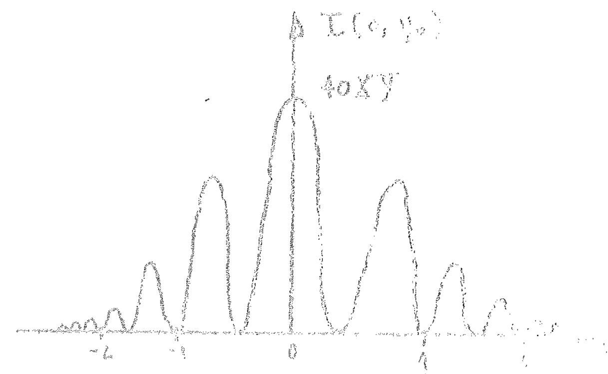
$$\boxed{I(x_0, y_0) = \frac{4X^2Y^2}{\lambda^2 z^2} \cos^2\left(\frac{\pi y_0 \Delta}{\lambda z}\right) \text{sinc}^2 \frac{X x_0}{\lambda z} \;\text{sinc}^2 \frac{Y y_0}{\lambda z}}$$

b. $\frac{X}{\lambda z} = 10 \qquad \frac{Y}{\lambda z} = 1 \qquad \frac{\Delta}{\lambda z} = \frac{3}{2}$

$$\Longrightarrow \quad I(x_0, y_0) = 40XY \cos^2\left(\frac{3\pi}{2} y_0\right) \text{sinc}^2\, 10x_0 \;\text{sinc}^2\, y_0$$



Nulls at Multiples of 0.1



4-4

a. $t(x_1, y_1) = \text{circ}\sqrt{x_1^2 + y_1^2}$

$$I(x_0=0, y_0=0) = I(0,0) = \frac{1}{\lambda^2 z^2}\left|\iint_{-\infty}^{\infty} \text{circ}\sqrt{x_1^2+y_1^2}\; e^{j\frac{k}{2z}(x_1^2+y_1^2)}\, dx_1\, dy_1\right|^2$$

$$I(0,0) = \frac{1}{\lambda^2 z^2}\left|\int_0^{2\pi} d\theta \int_0^1 r_1\, e^{j\frac{k}{2z}r_1^2}\, dr_1\right|^2$$

$$= \frac{4\pi^2}{\lambda^2 z^2}\left|\int_0^1 r_1\, e^{j\frac{k}{2z}r_1^2}\, dr_1\right|^2$$

Now make a change of variable

Let $u = r_1^2 \qquad du = 2r_1\, dr_1$

$$I(0,0) = \frac{4\pi^2}{\lambda^2 z^2}\left|\frac{1}{2}\int_0^1 e^{j\frac{k}{2z}u}\, du\right|^2$$

$$= \frac{\pi^2}{\lambda^2 z^2}\left[\frac{1}{j\frac{k}{2z}}\, e^{j\frac{k}{2z}u}\Big|_0^1\right]^2$$

$$= \left|e^{j\frac{k}{2z}} - 1\right|^2 = 2 - 2\cos\frac{k}{2z}$$

$$\boxed{I(0,0) = 2\left(1 - \cos\frac{k}{2z}\right)} = 4\sin^2\left(\frac{\pi}{2\lambda z}\right)$$

b. $t(x_1, y_1) = \begin{cases} 1 & a \le \sqrt{x_1^2+y_1^2} \le 1 \\ 0 & \text{otherwise} \end{cases}$

The only effect here is to change the limits of integration

$$I(0,0) = \frac{4\pi^2}{\lambda^2 z^2}\left|\int_a^1 r_1\, e^{j\frac{k}{2z}r_1^2}\, dr_1\right|^2$$

$$= \frac{4\pi^2}{\lambda^2 z^2}\left|\frac{1}{2}\int_{a^2}^1 e^{j\frac{k}{2z}u}\, du\right|^2$$

$$= \left|e^{j\frac{k}{2z}} - e^{j\frac{ka^2}{2z}}\right|^2$$

$$\therefore \boxed{I(0,0) = 2\left[1 - \cos\left\{\frac{k}{2z}(1-a^2)\right\}\right]} = 4\sin^2\left(\frac{k(1-a^2)}{4z}\right)$$

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4-5

From eq (4-29), two seperate spectral lines seperated by $\Delta\lambda$ in wavelength produce grating orders separated by $(q f_0 \Delta\lambda z)$. The width of a single order is $\frac{\lambda z}{l}$ from peak to first null. Thus the min^m resolvable wavelength difference $\Delta\lambda$ satisfies

$$q f_0 \Delta\lambda z = \frac{\lambda z}{l}$$

$$\text{or} \quad \frac{\lambda}{\Delta\lambda} = q f_0 l$$

But $f_0 l$ = # of full periods on the grating

$$\text{Thus} \quad \boxed{\frac{\lambda}{\Delta\lambda} = q M = q f_0 l}$$

While it is true that $J_q^2\left(\frac{m}{2}\right)$ falls off at very high q, the <u>fundamental limitation</u> preventing use of arbitrarily large q is the fact that the waves are <u>evanescent</u> and are rapidly attenuated.

4-6

<u>a.</u> $\tilde{t}(x,y) = \frac{1}{2}(1 + m\cos 2\pi f_0 x)$

$$\mathcal{F}\{\tilde{t}(x,y)\} = \frac{1}{2}\delta(f_x, f_y) + \frac{m}{4}\delta(f_x - f_0, f_y) + \frac{m}{4}\delta(f_x + f_0, f_y)$$

The transfer function [eq (4-11)] for Fresnel diffraction is: $\tilde{H}(f_x, f_y) = e^{jkz} e^{-j\pi\lambda z(f_x^2 + f_y^2)}$

Thus the spectrum of the Fresnel diffraction pattern associated with t is:

$$\mathcal{F}\{\tilde{U}(x_0,y_0)\} = \mathcal{F}\{\tilde{t}(x,y)\}\, H(f_x,f_y)$$

$$= e^{jkz}\left[\tfrac{1}{2}\delta(f_x,f_y) + \tfrac{m}{4}e^{-j\pi\lambda z f_0^2}\delta(f_x-f_0,f_y) + \tfrac{m}{4}e^{-j\pi\lambda z f_0^2}\delta(f_x+f_0,f_y)\right]$$

$$* \quad \tilde{U}(x_0,y_0) = e^{jkz}\left[\tfrac{1}{2} + \tfrac{m}{4}e^{-j\pi\lambda z f_0^2}e^{-j2\pi f_0 x} + \tfrac{m}{4}e^{-j\pi\lambda z f_0^2}e^{j2\pi f_0 x}\right]$$

$$I(x_0,y_0) = |\tilde{U}(x_0,y_0)|^2 = \left[\tfrac{1}{4} + \tfrac{m}{4}e^{j\pi\lambda z f_0^2}\cos 2\pi f_0 x + \tfrac{m}{4}e^{-j\pi\lambda z f_0^2}\cos 2\pi f_0 x + \tfrac{m^2}{4}\cos^2 2\pi f_0 x\right]$$

$$\boxed{I(x_0,y_0) = \tfrac{1}{4}\left(1+\tfrac{m^2}{2}\right) + \tfrac{m}{2}\cos 2\pi f_0 x \cos\pi\lambda z f_0^2 + \tfrac{m^2}{8}\cos 4\pi f_0 x}$$

b. (i) Going back to $*$ above for $\tilde{U}(x_0,y_0)$, we see that when $\pi\lambda z f_0^2 = N\pi$ or $z = \frac{N}{\lambda f_0^2}$ N integer, then

$$\tilde{U}(x_0,y_0) = e^{jkz}\left[1 \pm \tfrac{m}{2}\cos 2\pi f_0 x\right]$$ and we have pure Amplitute modulation

(ii) when $\pi\lambda z f_0^2 = (N-\frac{1}{2})\pi$ or $z = \frac{N-\frac{1}{2}}{\lambda f_0^2}$ $N \geq 0$, N integer

then $$\tilde{U}(x_0,y_0) = e^{jkz}\left[1 \pm j\tfrac{m}{2}\cos 2\pi f_0 x\right]$$

However $m \ll 1$, so

$$I(x_0,y_0) \approx 1 + \tfrac{m^2}{4}\cos^2 2\pi f_0 x \pm j\tfrac{m}{2}\cos 2\pi f_0 x \mp j\tfrac{m}{2}\cos 2\pi f_0 x$$

$$\approx 1 + \tfrac{m^2}{4}\cos^2 2\pi f_0 x \approx 1 \quad \text{since } m \ll 1$$

and so $I \approx$ constant, thus $\tilde{U}$ is approximately phase modulated.

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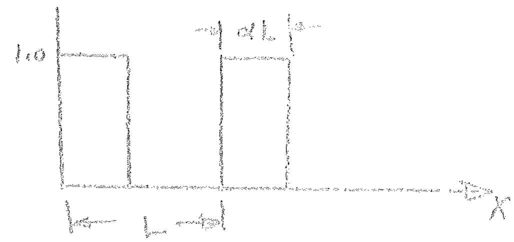
Problem 6



$$E(x,y) = \text{rect}\frac{x}{\alpha L} * \sum_{n=-\infty}^{\infty} \delta(x-nL, y)$$

$$= \text{rect}\frac{x}{\alpha L} * \left[\frac{1}{L}\text{comb}\left(\frac{x}{L}\right)\delta(y)\right]$$

$$\mathcal{F}\{E(x,y)\} = \left[\alpha L \,\text{sinc}\,\alpha L f_x \cdot \delta(f_y)\right] \text{comb}\,\frac{x}{L}$$

$$\mathcal{F}\{E(x,y)\} = \alpha \frac{\sin \pi\alpha L f_x}{\pi \alpha L f_x} \sum_{n=-\infty}^{\infty} \delta(f_x - \tfrac{n}{L}, f_y)$$

Now the $N=+1$ coefficient is:

$$C_{+1} = \alpha \frac{\sin \pi\alpha L\left(\frac{1}{L}\right)}{\pi\alpha L\left(\frac{1}{L}\right)} = \frac{\sin \pi\alpha}{\pi}$$

This maximizes when $\boxed{\alpha = \frac{1}{2}}$, yeild an $N=+1$ coefficient $\frac{1}{\pi}$

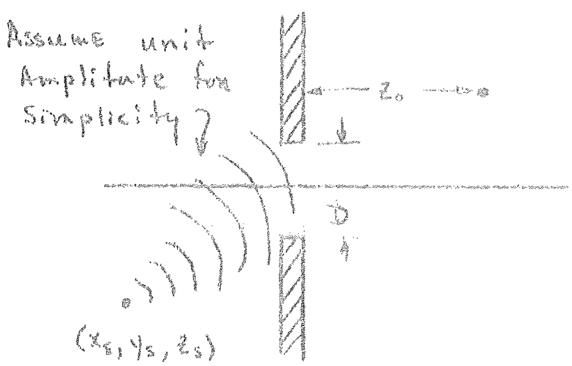
$$\text{max diffraction efficiency} = (C_{+1})^2 = \frac{1}{\pi^2} \approx \boxed{10.1\ \%}$$

The grating is more efficient since there is less absoption.

The problem is not well defined. One possible interpretation and solution is given below. (The Fraunhofer diffraction relation gives $\tilde{U}$ in the (x_0, y_0) plane as a function of $\tilde{U}$ in the (x_1, y_1) plane, provided that the condition $z_{01} \gg \frac{k}{2}(x_1^2 + y_1^2)_{max}$ is satisfied.

No assumptions are made here as to how $\tilde{U}(x_1, y_1)$ arises, the source behind (x_1, y_1) could be anywhere and the amplitute at (x_1, y_1) could be modified to produce some $\tilde{U}(x_0, y_0)$. The condition above must be satisfied, however.)

<u>A possible sol'n</u>



$$\tilde{U}(x_0, y_0) \approx \frac{e^{jkz_0}}{j\lambda z_0} e^{j\frac{\pi}{\lambda z_0}(x_0^2 + y_0^2)} \iint_{-\infty}^{\infty} t(x, y) \overbrace{e^{jkz_s} e^{j\frac{k}{2z_s}(x^2+y^2)}}^{\text{illuminating wave}} e^{j\frac{k}{2z_0}(x^2+y^2)} e^{-j\frac{2\pi}{\lambda z_0}(xx_0 + yy_0)} \, dx\,dy$$

Now if

$$\frac{k}{2}\left(\frac{1}{z_s} + \frac{1}{z_0}\right)(x^2+y^2)_{max} < \frac{\pi}{8}$$

or $\left(\frac{1}{z_s} + \frac{1}{z_0}\right)\frac{D^2}{4} < \frac{\lambda}{8}$

$$\boxed{\frac{z_s z_0}{z_s + z_0} > 2\frac{D^2}{\lambda}}$$

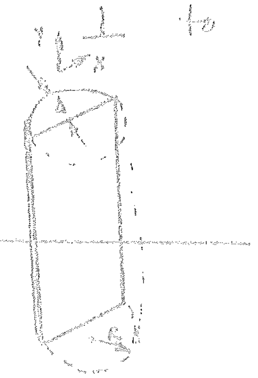
The Intergral becomes

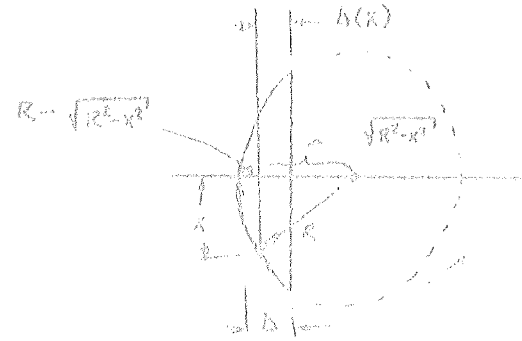
$$\tilde{U}(x_0, y_0) = \frac{e^{jk(z_0 + z_s)}}{j\lambda z_0} e^{j\frac{k}{2z_0}(x_0^2 + y_0^2)} \iint_{-\infty}^{\infty} t(x, y) e^{-j\frac{2\pi}{\lambda z_0}(x_0 x + y_0 y)} \, dx\,dy$$

Solutions

5-2

a. Assume the plane part of the lens is $\perp$ to the lens axis





$\Delta(x,y) = \Delta(x)$ a function of x only

$$\Delta(x) = \Delta - \left(R - \sqrt{R^2 - x^2}\right)$$

So $$\Delta(x) = \Delta - R\left(1 - \sqrt{1 - \left(\tfrac{x}{R}\right)^2}\right) \approx \Delta - R\left(1 - 1 + \tfrac{1}{2}\left(\tfrac{x}{R}\right)^2\right)$$

$$\Delta(x) = \Delta - \frac{x^2}{2R}$$

$$\therefore \quad \tilde{t}_\ell(x,y) = e^{jk\Delta}\, e^{jk(n-1)\Delta(x,y)} \qquad \text{eq } (5-1)$$

$$\tilde{t}_\ell(x,y) = e^{jk\Delta}\, e^{jk(n-1)\Delta}\, e^{-jk(n-1)\frac{x^2}{2R}}$$

$$\tilde{t}_\ell(x,y) = e^{jkn\Delta}\, e^{-jk(n-1)\frac{x^2}{2R}}$$

Now define $\frac{1}{f} = (n-1)\frac{1}{R}$

$$\boxed{\tilde{t}_\ell(x,y) = e^{jkn\Delta}\, e^{-j\frac{k}{2f}x^2}}$$

b. Lens focuses in the x direction while no focusing occurs in the y direction. The plane wave converges into a focal line at $x=0$, $z=f$.

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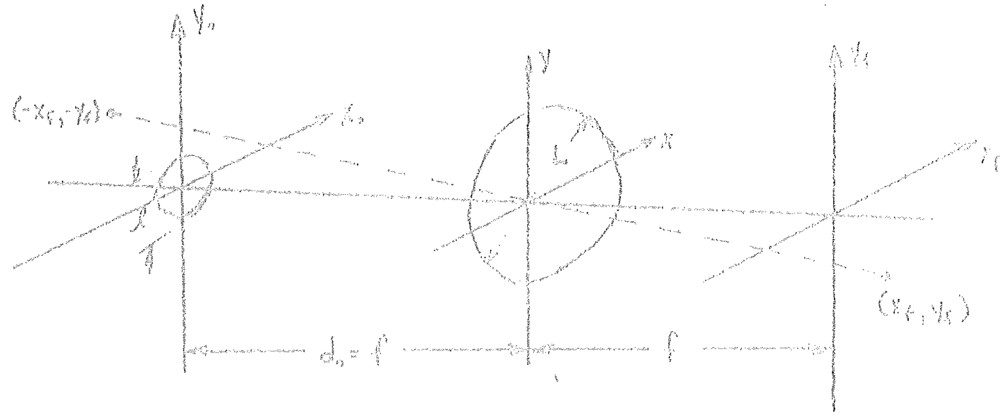
5-4



a. What is the largest value of x_f such that the lens projected back to the object plane includes all of the object. This occurs when

$$x_f - \frac{L}{2} \le -\frac{l}{2} \qquad x_f \le \frac{L-l}{2}$$

$$f_x = \frac{x_f}{\lambda f} \qquad \boxed{f_x \le \frac{L-l}{2\lambda f}}$$

b. $$f_x = \frac{(4-2)\,10^{-2}}{2\cdot 6\times10^{-7}\cdot .5} = \frac{1}{3}\cdot 10^{5}\ \frac{\text{cycles}}{\text{m}} = 33.3\ \frac{\text{cycles}}{\text{mm}}$$

$$\boxed{f_x = 33.3\ \frac{\text{cycles}}{\text{mm}}}$$

$L = 4\times10^{-2}$ m $f = 50\times10^{-2}$ m

$l = 2\times10^{-2}$ m $\lambda = 6\times10^{-7}$ m

c. The spectrum vanishes when the lens projection misses the object completely,

$$X_f \ge \frac{L+l}{2\lambda f}$$

$$\boxed{f_x \ge \frac{L+l}{2\lambda f} = 100\ \frac{\text{cycles}}{\text{mm}}}$$

EE 5360

5-5

$$\tilde{t}(r) = \frac{1}{2}\left(1 + \cos \alpha r^2\right) \mathrm{circ}\left(\frac{r}{2}\right)$$

$$\tilde{t}(r) = \left(\frac{1}{2} + \frac{1}{4} e^{j\alpha r^2} + \frac{1}{4} e^{-j\alpha r^2}\right) \mathrm{circ}\left(\frac{r}{2}\right)$$

$$\tilde{t}(r) = \tilde{t}_1(r) + \tilde{t}_2(r) + \tilde{t}_3(r)$$

where
$$\begin{cases} \tilde{t}_1(r) = \frac{1}{2}\,\mathrm{circ}\left(\frac{r}{2}\right) \\ \tilde{t}_2(r) = \frac{1}{4} e^{j\alpha r^2}\,\mathrm{circ}\left(\frac{r}{2}\right) \\ \tilde{t}_3(r) = \frac{1}{4} e^{-j\alpha r^2}\,\mathrm{circ}\left(\frac{r}{2}\right) \end{cases}$$

a. From Eq (5-10) we see that $\tilde{t}_2(r)$ may be viewed as the transmittance function of a diverging (negative) lens with pupil function $\mathrm{circ}\left(\frac{r}{2}\right)$; while $\tilde{t}_3(r)$ corresponds to a converging (positive) lens, also having pupil function $\mathrm{circ}\left(\frac{r}{2}\right)$. The term $\tilde{t}_1(r)$ gives rise to a plane wave (undiffracted).

b. The "diverging lens" has focal length $f_2 = -\frac{k}{2\alpha}$

The "converging lens" has focal length $f_3 = \frac{k}{2\alpha}$

hence the focal length is $\frac{k}{2\alpha} = |f_2| = |f_3|$

c. The fact that the focal lengths are wavelength dependent would hamper use of the screen for imaging polychromatic objects. The coefficients in $\tilde{t}_u(r)$ [illegible] also indicate an inefficiency of the screen for imaging.

EE 5560

5-8

$$I(x_f, y_f) = \frac{1}{\lambda^2 d^2}\left|\iint_{-\infty}^{\infty} t_o(x_o, y_o)\, P\left(x_o \frac{f}{d}, y_o \frac{f}{d}\right) e^{-j\frac{2\pi}{\lambda d}(x_o x_f + y_o y_f)}\, dx_o\, dy_o\right|^2$$

eq. [illegible]

On the x_f axis $y_f = 0$

The projected ~~radius~~ diameter of the pupil function ([illegible]) completly includes the object ($d_{max} = \sqrt{2}$), so we neglect the finite pupil size.

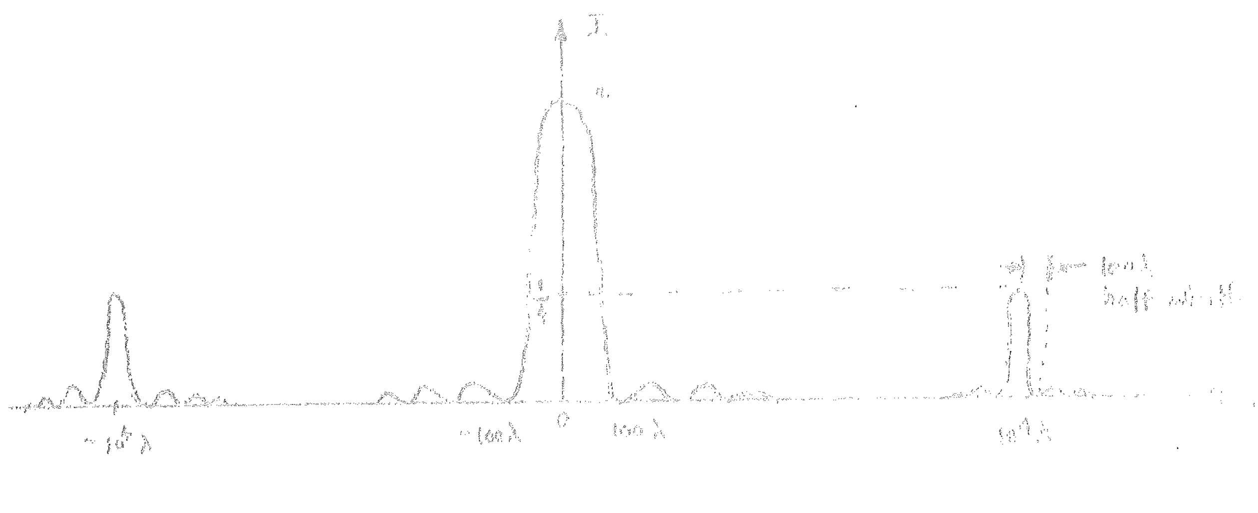
$$I(x_f, 0) = \frac{1}{\lambda^2 d^2}\left|\iint_{-\infty}^{\infty} (1 + \cos 2\pi f_o x_o)\, \mathrm{rect}\left(\frac{x_o}{L}\right) \mathrm{rect}\left(\frac{y_o}{L}\right) e^{-j\frac{2\pi}{\lambda d} x_o x_f}\, dx_o\, dy_o\right|^2$$

The result is the same as eq. (4-21) with $z = d$, $m = 1$, $y_o = 0$. Neglecting multiplicative constants:

$$I(x_f, 0) \propto \mathrm{sinc}^2\left(\frac{L x_f}{\lambda d}\right) + \frac{1}{4}\mathrm{sinc}^2\left[\frac{L}{\lambda d}(x_f + f_o \lambda d)\right] + \frac{1}{4}\mathrm{sinc}^2\left[\frac{L}{\lambda d}(x_f - f_o \lambda d)\right]$$

$d = 100$ cm $\quad f_o = 100 \frac{\text{cycles}}{\text{cm}} \quad L = 1$ cm

$$I(x_f, 0) \propto \mathrm{sinc}^2\left(\frac{x_f}{100\lambda}\right) + \frac{1}{4}\mathrm{sinc}^2\left[\frac{1}{100\lambda}(x_f + 10^4 \lambda)\right] + \frac{1}{4}\mathrm{sinc}^2\left[\frac{1}{100\lambda}(x_f - 10^4 \lambda)\right]$$



5-9

Let (x_Δ, y_Δ) be the coordinates across plane Δ in front of the focal plane.

Incident on the lens is:

$$U_l(x,y) = \text{rect}\frac{x}{\ell}\,\text{rect}\frac{y}{\ell}$$

so $$U_l'(x,y) = e^{-j\frac{\pi}{\lambda f}(x^2+y^2)}\,\text{rect}\frac{x}{\ell}\,\text{rect}\frac{y}{\ell}$$

(transmitted by the lens)

Incident on plane Δ we have:

$$U(x_\Delta, y_\Delta) = \frac{e^{j\frac{\pi}{\lambda(f-\Delta)}(x_\Delta^2+y_\Delta^2)}}{j\lambda(f-\Delta)}\iint_{-\infty}^{\infty}\text{rect}\frac{x}{\ell}\,\text{rect}\frac{y}{\ell}\,e^{-j\frac{\pi}{\lambda f}(x^2+y^2)}\,e^{j\frac{\pi}{\lambda(f-\Delta)}(x^2+y^2)}\,e^{-j\frac{2\pi}{\lambda(f-\Delta)}(x_\Delta x+y_\Delta y)}$$

We want the quadratic phase factor to vanish. Thus using one possible assumption:

$$-\frac{\pi}{\lambda f}+\frac{\pi}{\lambda(f-\Delta)}(x^2+y^2)_{max} < \frac{\pi}{8}$$

$$\frac{1}{\lambda}\left(\frac{1}{f-\Delta}-\frac{1}{f}\right)\left(\frac{\ell}{2}\right)^2 < \frac{1}{8}$$

$$\frac{\ell^2}{\lambda}\left(\frac{\Delta}{f(f-\Delta)}\right) < \frac{1}{2}$$

But $\frac{\Delta}{f(f-\Delta)} \approx \frac{\Delta}{f^2}$ so $$\boxed{\Delta < \frac{\lambda}{2}\left(\frac{f}{\ell}\right)^2}$$

or more accurately

$$\Delta < \frac{\lambda}{2\ell^2}(f^2-\Delta f)$$

$$\Delta\left(1+\frac{f\lambda}{2\ell^2}\right) < \frac{\lambda f^2}{2\ell^2}$$

$$\boxed{\Delta < \frac{f^2}{\frac{2\ell^2}{\lambda}+f}}$$

E E 5260 P.S. [illegible]

5-10

$$\tilde{t}(r) = \frac{1}{2}\left\{1 + \mathrm{Sqr}(\cos \alpha r^2)\right\} \mathrm{circ}\left(\frac{r}{l}\right)$$

$\mathrm{circ}\left(\frac{r}{l}\right)$ ~ pupil function, and may be neglected as far as finding focal lengths ($l >> r$)

Let $y \triangleq r^2$ and using hint realizing $X = \frac{2\pi}{\alpha}$

$$\tilde{t}(y) = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} \mathrm{sinc}\left(\frac{n}{2}\right) e^{jn\alpha y}\right] \mathrm{circ}\,\frac{\sqrt{y}}{l}$$

$$\Rightarrow \boxed{\tilde{t}(r) = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} \mathrm{sinc}\left(\frac{n}{2}\right) e^{jn\alpha r^2}\right] \mathrm{circ}\left(\frac{r}{l}\right)}$$

Each term represents a lens with different focal length. If f_n represents the focal length of the n^{th} lens, then

$$e^{-j\frac{\pi}{\lambda f_n}} = e^{jn\alpha r^2} \Rightarrow \frac{\pi}{\lambda f_n} = -n\alpha$$

$$\therefore \boxed{f_n = \frac{-\pi}{\lambda n \alpha}}$$

Relative energy brought to focus at n^{th} focal plane is $\frac{1}{4}\,\mathrm{sinc}^2\left(\frac{n}{2}\right)$

$\Rightarrow$ Zero order: Energy = 1/4, focal length = ∞

$\pm 2k^{th}$ order: Energy = 0, focal length = $\pm \frac{\pi}{[illegible]}$

$\pm(2k+1)$th order: Energy = $\frac{\mathrm{sinc}^2(k \pm \frac{1}{2})}{4}$, focal length = [illegible]

EE 5360 (JFW) Name: ______

Fall 1975

Quiz (10/21/75)

Open Book; 4 Problems

Work all problems on these sheets. Show your work and cite reference if appropriate. Budget your time! Point values are shown.

Prob. #1 (20 pts.): True-False Questions. Write "True" or "False" after each statement. (Scoring: +2 pts. if correct, 0 pts if left blank, -2 pts if incorrect).

1. The Fourier-Bessel transform is the 2-D F.T. of a circularly symmetric object. ______

2. One way to combat vignetting is to move the input transparency farther away from the Fourier transforming lens. ______

3. The input plane of a space-invariant optical system must be broken up into many iso'planatic patches. ______

Prob. #1 cont'd

④ In describing the transmittance function of a thin lens, we assumed that the lens imparts a spatially-varying phase delay, but imparts no attenuation to the incoming optical field

⑤ The angular spectrum of a diffracted light field is the product of the angular spectrum of the incident field with the angular spectrum transfer function for the diffracting aperture

⑥ A sinusoidal phase grating has a higher diffraction efficiency than a sinusoidal amplitude grating.

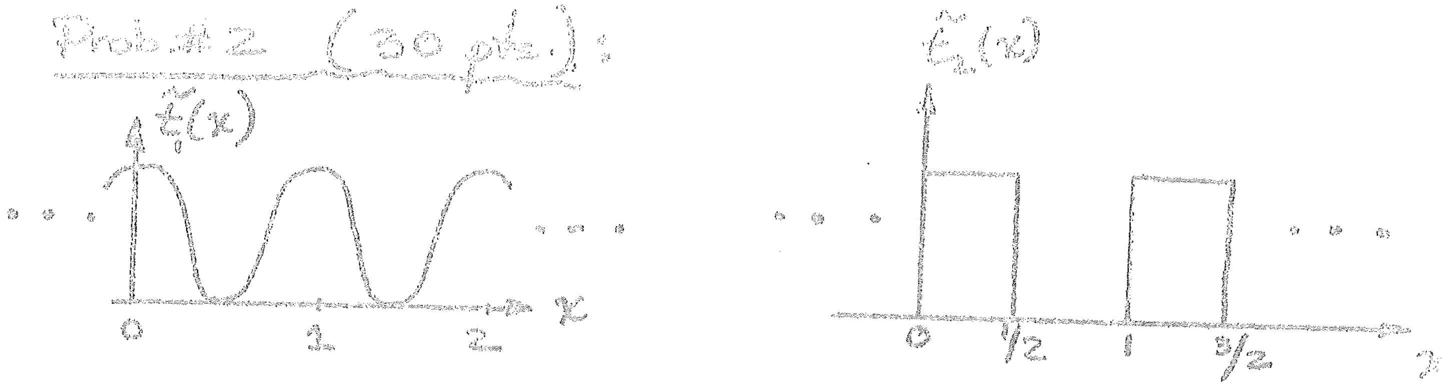
⑦ In the paraxial approx., one assumes that spherical wavefronts may be approximated by parabolic wavefronts

⑧ Fresnel diffraction may be viewed as a special case of Fraunhofer diffraction

⑨ The comb(x) and $e^{-\alpha x^2}$ functions both have the property that each has a F.T. having the same form as the function.

⑩ In order to increase the size of the Fourier spectrum (actually the power spectrum when viewed) obtained with a thin lens, we decrease λ and/or f.

Prob. # 2 (30 pts.):



Two amplitude gratings are shown above. (one sinusoidal, one square-wave). Compare these gratings quantitatively with respect to

ⓐ The percentage of incident light intensity absorbed by the grating

ⓑ The percentage of incident light intensity appearing in the zero-order component.

ⓒ The percentage of incident light intensity appearing in a single first-order component.

Hint: $\text{comb}(x) * \text{rect}(2x)$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{\sin(n\pi/2)}{\pi n}\right] e^{j2\pi n x}$$

Prob. # 3 (30 pts)

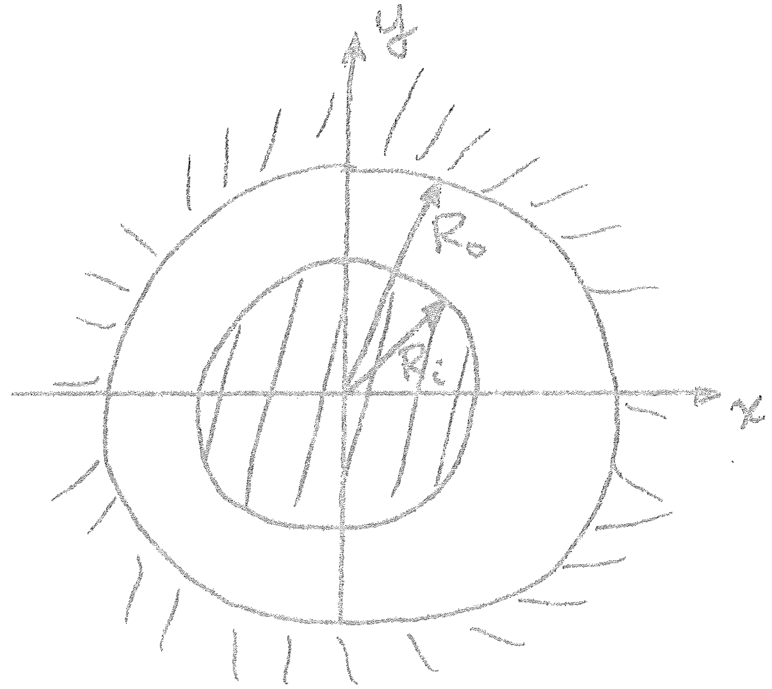
ⓐ (15 pts). Find the diameter of the central bright disk in the focal plane of a telescope lens having a diameter of 1 meter, as it looks at a distant star. Assume $f = 15$ meters, $\lambda = 5500$ Å (1 Å $= 10^{-10}$ meters)

ⓑ (15 pts). A sinusoidal amplitude grating has a 2000 cycles/mm ruling. How wide must the grating be to be able to resolve the modes of a 6328 Å He-Ne laser, assuming the modes are separated by 500 MHz? (Recall that $c = 3 \times 10^8$ m/sec.). (assume cycles/mm = lines/mm)

Prob. # 4 (20 pts): Evaluate the Fraunhofer diffraction pattern of the aperture shown below. Sketch the cross section of the pattern.

$R_i = 2$ mm

$R_o = 3$ mm



EE [illegible] Name:

Fall 1975

FINAL EXAM (12/17/75)

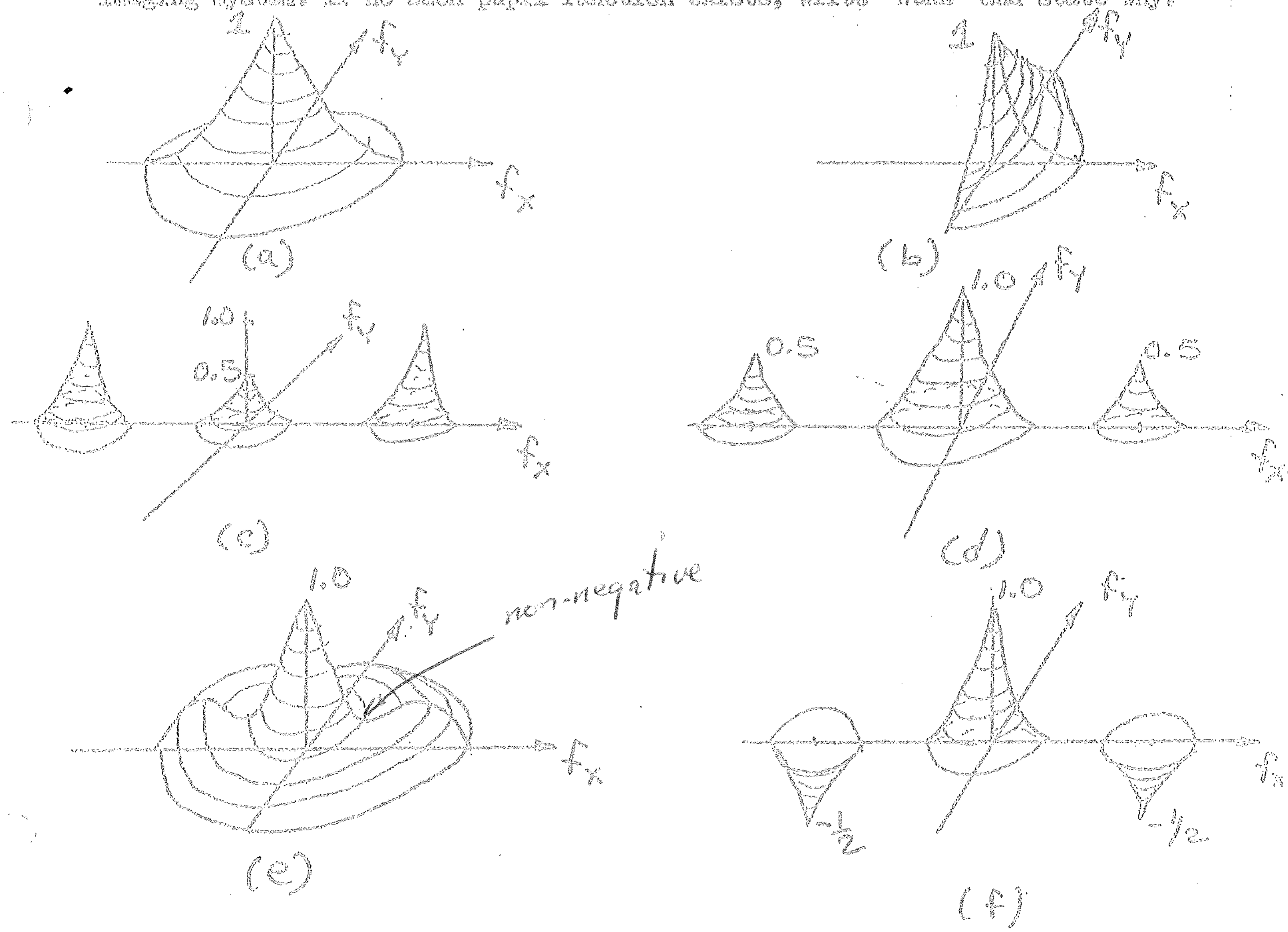
1	
2	
3	
4	
5	

Open Book

200 pts. total

Do all problems. Start each problem on a new sheet of paper. Show your work. The point values are shown, so budget your time!

Problem 1: (40 pts.) Shown below are several 2-dimensional potential OTF's. In each case sketch a black and white (i.e. opaque or clear) pupil function that could generate an OTF of the specified shape in an aberration-free imaging system. If no such pupil function exists, write "NONE" and state why.



PROBLEM II (40 pts.) A diffraction-limited circular lens of diameter 2 cm. and focal length 5 cm. is available. It is to be used in a coherent imaging system (λ = 5000 Å) to determine the spatial frequency of a certain sinusoidal amplitude grating with amplitude transmittance

$$\tilde{t}_1(x,y) = \frac{1}{2}\left[1 + \cos\left(2\pi f_0 x\right)\right]$$

The grating frequency is known to be about 2000 cycles/ cm., but is not known precisely.

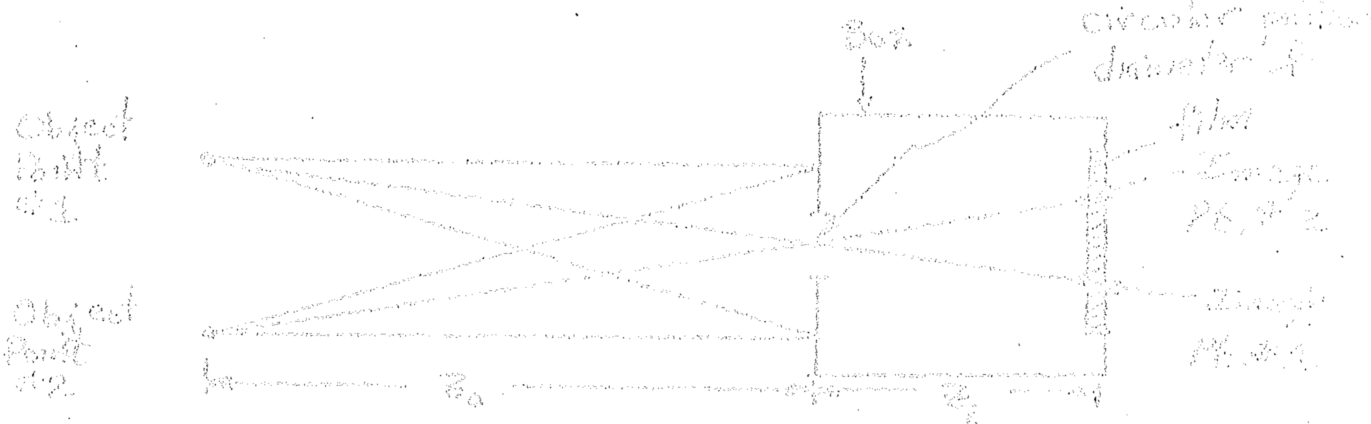
(20) (a) What is the coherent cutoff frequency of the imaging system assuming that the grating (i.e. the object) is placed 10 cm. in front of the lens. Sketch (and label the sketch) the distribution of intensity that you would expect to see in the image plane in this case.

(20) (b) Assume that a second grating of known frequency (2000 cycles/mm) is also available; its amplitude transmittance may be taken to be

$$\tilde{t}_2(x,y) = \frac{1}{2}\left[1 + \cos\left(4000\pi x\right)\right]$$

where x is in mm. Explain how the known grating and the available imaging system can be used to measure f_0. Specify any limitations on f_0 that seem appropriate.

Problem [illegible]. (40 pts.) A "pinhole camera" is illustrated in the figure [illegible]. The simple object point sources illustrate how the camera works.



You may make the following assumptions:

(1) The pinhole is circular, and has diameter ℓ.

(2) The object distance z_o is infinite.

(3) The object is incoherent and quasi-monochromatic, with mean wavelength λ.

(4) Only small angles are involved.

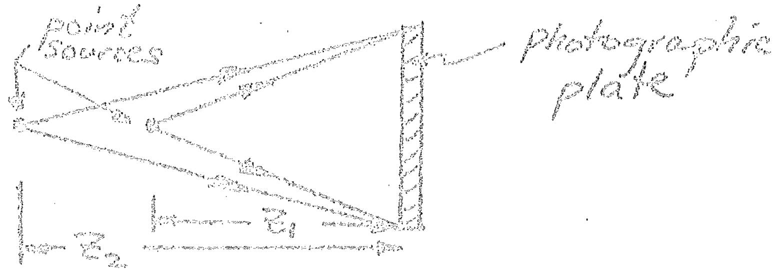
---

(15) (a) Using a basic definition of the OTF as the normalized transfer function of the incoherent imaging system, find the [illegible] OTF of this imaging system under the assumption that the pinhole diameter ℓ is so <u>small</u> as to place the image plane in the region of Fraunhofer diffraction. (<u>hint</u>: first find the point spread function).

(15) (b) Find the OTF of the system when the pinhole diameter is so <u>large</u> as to place the image plane deep within the region of Fresnel diffraction, where geometrical optics yields accurate answers.

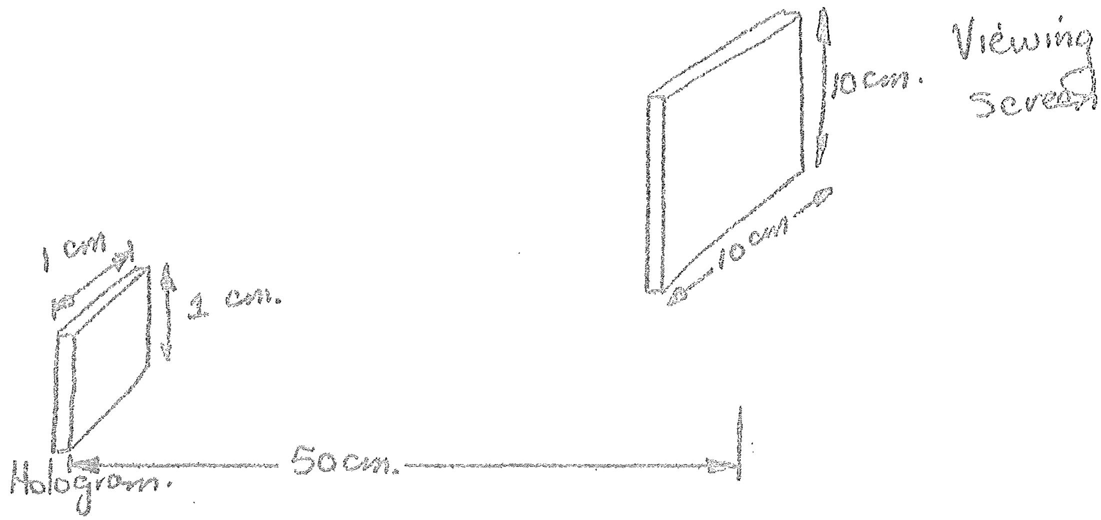
(10) (c) Defining the cutoff [illegible] frequency of the OTF as the frequency where the response has its first zero, predict on the basis of the results of (a) and (b) the pinhole size which yields the <u>highest</u> cutoff frequency.

[illegible] ([illegible]) A so-called "Fresnel zone plate" is formed by recording the pattern of interference between two divergent waves, as shown below:



The film is developed in such a way as to yield an amplitude transmittance proportional to exposure. What is the focal length of the resulting transparency? (i.e. the resulting Fresnel lens).

Problem 5: (40 pts.) In a certain holographic display device, it is desired to cast an image of a transparency on a viewing screen. The object is a square transparency 1 cm. by 1 cm. in size. The image on the screen is to be 10 cm. by 10 cm. in size, and the screen is to be 50 cm. from the hologram (see figure below).



The same wavelength is used for both recording and reconstruction.

(15) (a) How far from the hologram should the object transparency be placed during the recording process?

(25) (b) Describe the reference and reconstruction illuminations necessary to achieve the desired image. If more than one soln. exists, state the various alternatives.

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STAT. OPTICS
EE 5358

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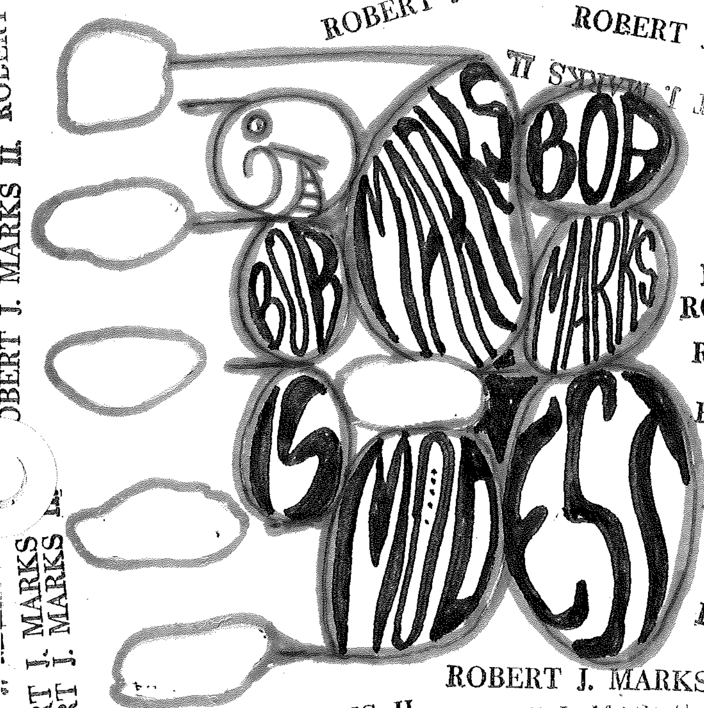

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1-19-76 (MON)

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NEED FOR STATISTICS IN OPTICS

(1) OPTICAL WAVES ARE, IN GENERAL, RANDOM PROCESSES ⇒ STATISTICAL PROPERTIES ARE IMPORTANT IN IMAGING, & OTHER OPTICAL MEASUREMENTS.

(2) THE OBJECT SCENES OF INTEREST ARE STATISTICAL IN NATURE. IF THEY WEREN'T, WE COULD SPECIFY THEM A PRIORI & WOULD HAVE NO NEED TO FORM IMAGES.

(3) AN IMAGING SYSTEM MAY HAVE RANDOM PROPS (TURBULANCE, RANDOM GRINDING, POLISHING ERRORS, ETC.)

(4) THE DETECTION PROCESS INTRODUCE RANDOM FLUNCTUATIONS WHICH OBSCURE THE DETAILS OF INTEREST (FILM GRAIN NOISE, SHOT NOISE, PHOTON STATISTICS).

(5) CHARACTERIZING THE OBSERVER'S INTERPRETATION OF THE SCENE (DETECTION AND ESTIMATION THEORY).

REVIEW OF STATISTICAL CONCEPTS

RANDOM VARIABLES

THINK OF A RANDOM EXPERIMENT WITH POSSIBLE NUMERICAL OUTCOMES $X_1, X_2, \ldots$. LET n_K BE THE NUMBER OF TIMES X_K OCCURS IN N TRIALS.

$$\lim_{N \to \infty} \frac{n_K}{N} = \text{RELATIVE FREQUENCY OF OCCURANCE OF } X_K$$

$$= P(X_K)$$

$$= P[X_K \text{ OCCURS IN ANY ONE TRIAL (ASSUMING INDEPENDENT TRIALS)}]$$

$X_1, X_2, \ldots$ ARE POSSIBLE VALUES

(EX) ROLL OF A DIE

$$\{X_K\} = \{1, 2, 3, 4, 5, 6\}$$

LET $\underline{X}$ = RANDOM VARIABLE WHICH CAN TAKE ON POSSIBLE VALUES $\{X_K\}$

$$P_{\underline{X}}(1) = \frac{1}{6},\quad P_{\underline{X}}(2) = \frac{1}{6},\quad \text{ETC.}$$

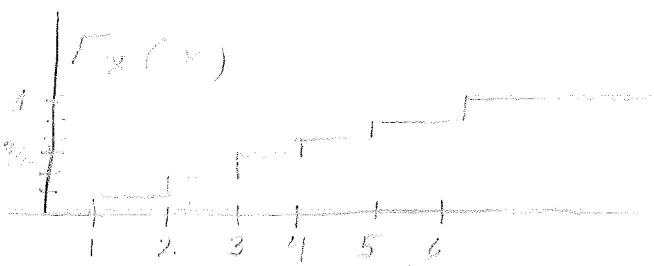
1-21-76

$P_X(x_k) \quad \lim_{N\to\infty} \frac{n_k}{N}$

DISTRIBUTION FUNCTION

$F_X(x) \triangleq P(X \le x)$

EX. FAIR DIE

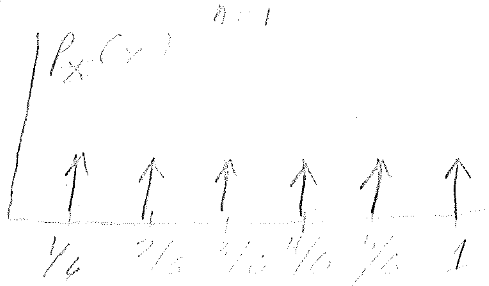


PROBABILITY DENSITY FCNS.

$$p_X(x) = \frac{d}{dx} F_X(x)$$

EX FOR FAIR DIE

$$p_X(x) = \sum_{n=1}^{6} \frac{1}{6}\,\delta(x-n)$$



$$\int_{-\infty}^{\infty} p_X(x)\,dx = 1$$

$$F_X(x) = \int_{-\infty}^{x} p_X(x)\,dx$$

$$p_X(x) = P[x - dx < X \le x]$$

$$P[a \le x \le b] = F_X(b) - F_X(a) = \int_a^b p_X(x)\,dx$$

JOINTLY DISTRIBUTED RANDOM VARIABLES

$$F_{XY}(x,y) = P[X \le x, Y \le y]$$

EX TWO COIN FLIPS

X	Y	PROB
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

$$F_{XY}(0,1) = 1/2$$

JOINT DENSITY FUNCTION

$$p_{XY}(x,y) \quad \frac{d^2}{dx\,dy} F_{XY}(x,y)$$

CONDITIONAL STATISTICS

$$F_{XY}(x,y) = F_Y(y)\, F_{X|Y}(x|y)$$
$$= F_X(x)\, F_{Y|X}(y|x)$$
$$= P[X \le x]\, P[Y \le y \mid X \le x]$$

$$p_{XY}(x,y) = \frac{d^2}{dx\,dy} F_{XY}(x,y)$$

$$= p_{X|Y}(x|y)\, p_Y(y)$$

$$= p_X(x)\, p_{Y|X}(y|x)$$

$p_Y(y)$ = MARGINAL DENSITY FOR Y
$p_X(y)$ = MARGINAL DENSITY FOR X

STATISTICAL INDEPENDENCE

WE SAY THAT X AND Y ARE STATISTICALLY INDEP. RANDOM VARIABLES IFF $p_{X|Y}(x|y) = p_X(x)$ AND $p_{Y|X}(y|x) = p_Y(y)$

$\Rightarrow p_{XY}(x,y) = p_X(x)\,p_Y(y)$

AVERAGES (MOMENTS)

$\bar{X}$ = MEAN VALUE OF X = $E[X]$

$= \int_{-\infty}^{\infty} x\, p_X(x)\,dx$ = FIRST MOMENT

SAY $Y = f(X)$. THEN

$\bar{Y} = \int_{-\infty}^{\infty} y\, p_Y(y)\,dy = \int_{-\infty}^{\infty} f(x)\, p_X(x)\,dx$

$= E[f(x)]$

VARIANCE: $\sigma_X^2 = E[(X-\bar{X})^2]$

$= \overline{X^2} - \bar{X}^2$

= SECOND CENTRAL MOMENT

σ_X = STANDARD DEVIATION

n^{th} CENTRAL MOMENT:

$E[(X-\bar{X})^n] = \int_{-\infty}^{\infty} (x-\bar{x})^n p_X(x)\,dx$

CORRELATION COEFFICIENT FOR X & Y

$$\rho = \frac{\overline{XY} - \bar{X}\bar{Y}}{\sigma_X \sigma_Y}$$

IF X & Y ARE INDEPENDENT, $\rho = 0$
CONVERSE NOT NECESSARILY TRUE EXCEPT FOR GAUSSIAN RANDOM VARIABLES.

1-22-75 (FRI)

CHARACTERISTIC FUNCTION:

$$M_X(jv) \triangleq E\left[e^{jvX}\right]$$

$$= \int_{-\infty}^{\infty} e^{jvx} p_X(x)\, dx$$

$$= \mathcal{F}^{-1}\left[p_X(x)\right]$$

$$\frac{d^n M_X(jv)}{dv^n} = (j)^n \int_{-\infty}^{\infty} x^n e^{jvx} p_X(x)\, dx$$

$$\Rightarrow (-j)^n \left.\frac{d^n M_X(jv)}{dv^n}\right|_{v=0} = E\left[x^n\right]$$

$$\text{AND} \quad M_X(jv) = 1 + \sum_{k=1}^{\infty} \frac{\overline{x^k}}{k!}(jv)^k$$

FOR JOINT DISTRIBUTIONS:

$$M_{XY}(jv_1, jv_2) = \overline{e^{jv_1 x + jv_2 y}}$$

$$= \int_{-\infty}^{\infty}\int e^{j(v_1 x + v_2 y)} p_{XY}(x,y)\, dx\, dy$$

$$M_{XY}(jv_1, 0) = M_X(jv_1)$$
$$M_{XY}(0, jv_2) = M_Y(jv_2)$$

FOR STATISTICALLY INDEPENDENT X & Y

$$M_{XY}(jv_1, jv_2) = M_X(jv_1)\, M_Y(jv_2)$$

$$\overline{x^n y^k} = (-j)^{n+k} \left.\frac{\delta^{n+k} M_{XY}(jv_1, jv_2)}{\delta v_1^n\, \delta v_2^k}\right|_{v_1 = v_2 = 0}$$

EXAMPLES:

1. GAUSSIAN R.V.:
$$M_X(jv) = \exp\left(jv\bar{X} - \frac{v^2\sigma_X^2}{2}\right)$$

2. NEG. EXPONENTIAL (ONE SIDED)
$$M_X(jv) = 1/(1 - j\bar{X}v)$$

SUM OF TWO RANDOM VARIABLES:

IF $Z = X + Y$

$$M_Z(jv) = \int_{-\infty}^{\infty} e^{jv(x+y)} p_{XY}(x,y)\,dx\,dy$$

FOR STATISTICAL INDEPENDENT X & Y:

$$M_Z(jv) = M_X(jv)\,M_Y(jv)$$

$$\Rightarrow p_Z(z) = p_X(x) * p_Y(y) \Leftarrow \text{CONVOLUTION}$$

$$= \int_{-\infty}^{\infty} p_X(\xi)\,p_Y(z-\xi)\,d\xi$$

CENTRAL LIMIT THEOREM

(SEE GNEDENKO & KOLMOGOROV)

IF $Z = X_1 + X_2 + \dots + X_N$ AND THE X_i, $i = 1, 2, \dots, N$ ARE STATISTICAL INDEPENDENT, THEN, UNDER RATHER GENERAL CONDITIONS, ONE CAN SHOW THAT

$$\lim_{N\to\infty} p_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_Z} e^{-\frac{(z-\bar{z})^2}{2\sigma_Z^2}}$$

$$\bar{Z} = \bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_N$$

$$\sigma_Z^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_N^2$$

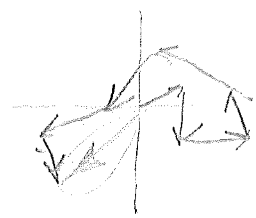
BASIC REQUIREMENT: NONE OF THE RANDOM VARIABLES DOMINATE THE STATISTICS OF Z_n (i.e., ALL THE σ_{x_i} ARE OF SAME MAGNITUDE)

ASIDE:

RANDOM PHASOR SUM: $Z_i = R_i e^{j\theta_i}$

$$\sum_{i=1}^{N} Z_i = \sum_{i=1}^{N} R_i e^{j\theta_i} = Z$$

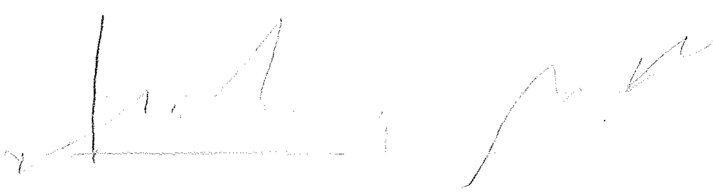
CLASSICAL BROWNIAN RANDOM WALK PROBLEM:



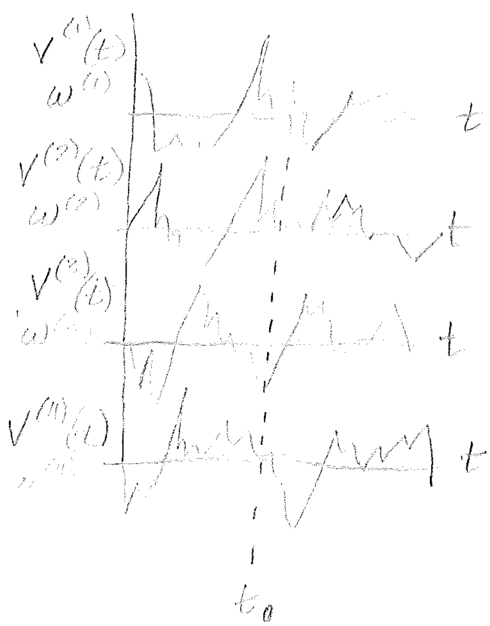
$R \sim$ RAYLEIGH

$\theta \sim$ UNIFORM

RANDOM PROCESSES:



SAMPLE FUNCTION FROM AN ENSEMBLE OF POSSIBLE FCNS.



$\{x(t)\}$ = RANDOM PROCESS ENSEMBLE

$X(t_0)$ IS A NUMBER OF RANDOM VARIABLES.

TO COMPLETELY SPECIFY, NEED JOINT PROBABILITY

AUTOCORRELATION

$$R_x(t_1,t_2) = E[X(t_1), X(t_2)]$$
$$= \int_{-\infty}^{\infty}\int x(t_1)\,x(t_2)\,p_{x_1,x_2}(x_1,x_2;t_1,t_2)\,dx_1\,dx_2$$

IF PROCESS IS WIDE SENSE STATIONARY

$$R_x(t_1,t_2) = R_x(t_2-t_1) = R_x(t)$$

ERGODICITY:

ALL TIME AVERAGES ARE IDENTICAL TO THE ANALOGOUS ENSEMBLE AVERAGE.

$$E[x(t)] = \overline{x(t)} = \int_{-\infty}^{\infty} x(t)\,p_x(x;t)\,dx$$

TIME AVERAGE:

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t')\,dt'$$

SIMILARLY

$$\langle x^n(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^n(t)\,dt$$

FOR ERGODIC SYSTEM:

$$\overline{x^n(t)} = \langle x^n(t)\rangle$$

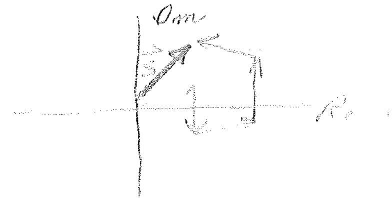
CAN'T BE NON-STATIONARY AND NOT ERGODIC.

1-26-75 (MON)

RANDOM PHASOR SUMS

$$\tilde{S} = \sum_{k=1}^{N} A_k e^{j\phi_k} = R e^{j\theta} = S_1 + j S_2$$

(DRUNKARD'S WALK)



$\{A_k\}$ $\{\phi_k\}$ ARE RANDOM VARIABLES

N IS LARGE

ASSUME

(1) INDEPENDENT PHASORS

(2) ALL CONTRIBUTIONS HAVE COMPARABLE MEAN SQUARE VALUES

$$S_1 = \mathrm{Re}\,\tilde{S} = \sum_{k=1}^{N} A_k \sin\phi_k$$
$$S_2 = \mathrm{Im}\,\tilde{S} = \sum_{k=1}^{N} A_k \sin\phi_k$$

BY CENTRAL LIMIT THEOREM, S_1 & S_2 CAN BE APPROXIMATED AS CRV (GAUSSIAN RANDOM VARIABLES). S_1 AND S_2 ARE CHARACTERIZED BY $\overline{S_1}$, $\overline{S_2}$, $\overline{S_1^2}$, $\overline{S_2^2}$, $\overline{S_1 S_2}$

$$\mathrm{cov}[S_1, S_2] \triangleq E[(S_1 - \overline{S_1})(S_2 - \overline{S_2})]$$

ASSUME $\forall$ K, A_k, & ϕ_k ARE STAT. INDEPEN.

THE ϕ_k'S ARE UNIFORMLY DISTRIBUTED, ON $[0, 2\pi]$.

$$\overline{S_1} = \overline{\sum_{K=1}^{N} A_K \cos\phi_K} = \sum_{K=1}^{N} \overline{A_K \cos\phi_K}$$

BY STATISTICAL INDEPENDENCE:

$$\overline{S_1} = \sum_{K=1}^{N} \overline{A_K}\ \overline{\cos\phi_K}$$

BUT $\overline{\cos\phi_K} = 0 \Rightarrow \overline{S_1} = 0$

SIMILARLY

$$\overline{S_2} = \sum_{K=1}^{N} \overline{A_K}\ \overline{\sin\phi_K} = 0$$

AND

$$\overline{S_1^2} = \sum_{K=1}^{N}\sum_{l=1}^{N} \overline{A_K A_l}\ \overline{\cos\phi_K \cos\phi_l}$$

$$\overline{\cos\phi_K \cos\phi_l} = \begin{cases} \overline{\cos\phi_K}\ \overline{\cos\phi_l} = 0 & \forall K \neq l \\ \overline{\cos^2\phi_K} & ; K = l \end{cases}$$

$$\overline{\cos^2\phi_K} \int_0^{2\pi} \cos^2\phi_K\, P_{\phi_K}(\phi_K)\, d\phi_K)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2\phi_K\, d\phi_K = \frac{1}{2}$$

$$\therefore \overline{\cos\phi_K \cos\phi_l} = \begin{cases} 0 & ; K \neq l \\ \frac{1}{2} & ; K = l \end{cases}$$

GIVES

$$\overline{S_1^2} = \sum_{K=1}^{N} \frac{1}{2}\overline{A_K^2} \quad ; \quad \overline{S_2^2} = \sum_{K=1}^{N} \frac{1}{2}\overline{A_K^2}$$

$$= \sigma_{S_1}^2 \qquad\qquad = \sigma_{S_2}^2$$

$$\overline{S_1 S_2} = \sum_{K=1}^{N}\sum_{l=1}^{N} \overline{A_K A_l}\ \overline{\cos\phi_K \sin\phi_l}$$

$$\overline{\cos\phi_K \sin\phi_l} \begin{cases} 0 & ; l \neq K \\ \frac{1}{2}\overline{\sin 2\phi_K} & ; K = l \end{cases}$$

$$= 0$$

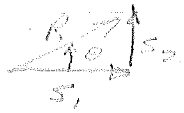
THUS

$\overline{S_1 S_2} = 0 \Rightarrow S_1 \,\&\, S_2$ ARE UNCORRELATED GRV'S

$\therefore$ STAT. INDEPENDENT

$$p_{s_1 s_2}(s_1, s_2) = \frac{1}{2\pi\sigma^2} e^{\frac{-(s_1^2 + s_2^2)}{2\sigma^2}}$$

WHERE $\quad \sigma^2 = \sum_{k=1}^{N} A_k^2/2$



$$r = |R| = \sqrt{s_1^2 + s_2^2}$$

$$\theta = \operatorname{atan}(s_2/s_1)$$

$$s_1 = r\cos\theta$$

$$s_2 = r\sin\theta$$

MUST DO THE TRANSFORMATION OF RANDOM VARIABLES FROM s_1, s_2 TO R, θ.

$|J|$ = JACOBIAN OF TRANSFORMATION

$$= \begin{vmatrix} \frac{\delta s_1}{\delta r} & \frac{\delta s_2}{\delta r} \\ \frac{\delta s_1}{\delta \theta} & \frac{\delta s_2}{\delta \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\Rightarrow |J| = r$$

$$p_R(r,\theta) = p_{s_1 s_2}(r\cos\theta, r\sin\theta)\,|J|$$

$$= p_{s_1 s_2}(r\cos\theta, r\sin\theta)\,|J|$$

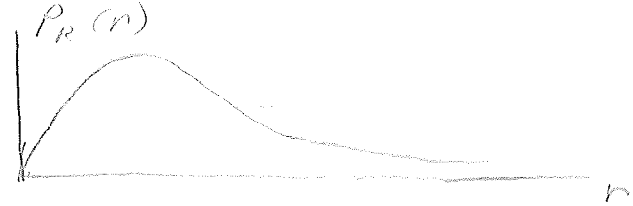
$$= \begin{cases} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} & ; 0 < r < \infty;\ 0 < \theta < 2\pi \\ 0 & ; \text{OTHERWISE} \end{cases}$$

WHAT ARE THE MARGINAL DENSITIES

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)\,d\theta$$

$$= \begin{cases} \dfrac{r}{\sigma^2} e^{-r^2/2\sigma^2} & ; \ 0 \le r < \infty \\ 0 & ; \ r < 0 \end{cases}$$

= RALEIGH DISTRIBUTION



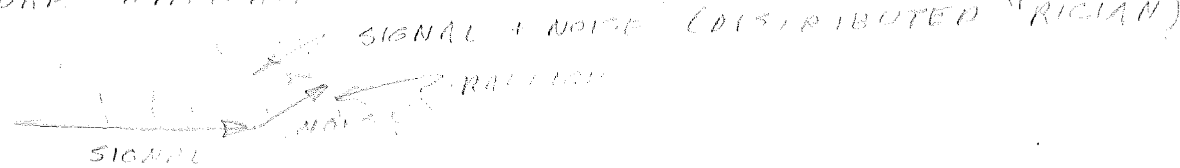
$$p_\Theta(\theta) = \int_0^{\infty} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\,dr$$

$$= \begin{cases} \dfrac{1}{2\pi} & ; \ 0 \le \theta \le 2\pi \\ 0 & ; \ \text{OTHERWISE} \end{cases}$$

NOTE THAT: $p_{R,\Theta}(r,\theta) = p_R(r) \cdot p_\Theta(\theta)$

$\Rightarrow$ r AND Θ ARE STATISTICALLY INDEPENDENT.

RADAR APPLICATION



1-28-76 (WED) (HANDOUT)

$v(t) = A\cos[2\pi\nu_0 t + \phi]$

$= Re[A e^{j(2\pi\nu_0 t + \phi)}]$

$= A e^{j\phi} e^{j2\pi\nu_0 t}$ ← MONOCHROMATIC

$\underline{v}(t) = \underbrace{A e^{i\phi}}_{A_0} e^{j2\pi\nu_0 t}$

ANALYTIC SIGNAL → SUPPRESS NEGATIVE FREQ, DOUBLE POS FREQ

$\underline{v}(t) \triangleq 2\int_0^{\infty} \underline{V}(\nu) e^{j2\pi\nu t} d\nu$

1-30-76 (FRI)

STRUCTURE FUNCTION

$D_v(t_1, t_2) = \overline{[v(t_1) - v(t_2)]^2}$ WHERE $v(t)$ IS A REAL VALUED RANDOM PROCESS

2-2-76

2-6-76

$D_\theta(t_2, t_1) = \overline{[\theta(t_2) - \theta(t_1)]^2}$

$= \overline{[2\pi \int_{-\infty}^{t_2} \nu_R(\xi) d\xi - 2\pi\int \nu_r(\xi) d\xi]^2}$

$= 4\pi^2 \overline{[\int_{t_1}^{t_2} \nu_R(\xi) d\xi]^2}$

$= 4\pi \overline{[\int_{-\infty}^{\infty} rect\left(\frac{\xi - (t_1+t_2)/2}{t_2 - t_1}\right) \nu_r(\xi) d\xi]^2}$

$= 4\pi^2 \iint rect\left[\frac{\xi - (t_1+t_2)/2}{t_2 - t_1}\right] rect\left[\frac{\eta - (t_1+t_2)/2}{t_1 - t_2}\right] \overline{\nu_R(\xi)\,\nu_R(\eta)}\, d\xi\, d\xi$

$= 4\pi^2 \iint rect\left[\frac{\xi - (t_1+t_2)/2}{\tau = t_2 - t_1}\right] rect\left(\frac{\eta - (t_1+t_2)/2}{\tau = t_1 - t_2}\right) \times \Gamma_\nu(\xi - \eta)\, d\xi\, d\eta$

2-9-76 (MON)

3-10-76

2 3 BASIC PAPERS

8/10 PG. PAPER

20 MIN ILLUS. TALK

INTERFEROMETRIC STAR TRACKING

AJIT QUATRA

SPECKLE INTERFEROMETRY

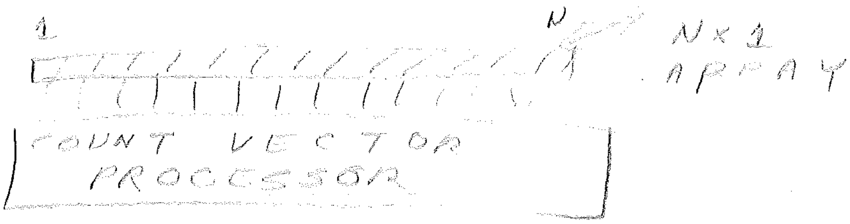
$$\langle |I(f)|^2 \rangle = |O(f)|^2 \langle |T(f)|^2 \rangle$$

TURBULANCE IN THE ATMOSPHERE

TALKS

75 TO 90

5/3/76



$\underline{n} \Leftarrow N \times 1$ VECTOR

$$\underline{n} = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix}$$

$\overline{\underline{n}}$ = MEAN OF $\underline{n}$

$\overline{\underline{n}} = D\underline{m} + \underline{b}$ ← BACKGROUND RATE (KNOWN)

$N \times M$ DEGRADATION MATRIX ($N \times m$)

(ANALAGOUS TO $i = o * h + n$)

$\underline{m}$ = IDEAL IMAGE

ADDITIVE NOISE: $\underline{n} = D\underline{m} + \underline{\epsilon}$

$\hat{\underline{m}} = R(\underline{n} - \underline{b})$

R = RESTORATION MATRIX ⇒ $R = M \times N$

LOOK @ LINEAR UNBIASED ESTIMATES

UNBIASED ⇒ $\overline{\hat{\underline{m}}} = \underline{m}$

NASC FOR UNBIASED IS $RD = I_{MM}$

POISSON MODEL

$\underline{n}$ ~ POISSON

$\underline{\bar{n}} = Dm + b$

~~ADDITIVE~~ (SIGNAL DEPENDENT) MODEL

~~$\underline{n} = Dm + \epsilon$~~

$E[(n-\bar{n})(n-\bar{n})'|\underline{m}] = B$ = CONDITIONAL COVARIANCE

$$= \begin{bmatrix} \bar{n}_1 & & & 0 \\ & \bar{n}_2 & & \\ & & \ddots & \\ 0 & & & \bar{n}_N \end{bmatrix}$$

ADDITIVE (SIGNAL-INDEPENDENT) MODEL

$B = \sigma^2 I_N \qquad \ni \underline{\bar{\epsilon}} = b$

$$\underline{n} = D\underline{m} + \underline{\epsilon}$$

IN POISSON MODEL, ASSUME $\underline{m}$ HAS MEAN m_0

MEAN SQUARE ERRORS

$E\|\hat{\underline{m}} - \underline{m}\|^2 = b\,Tr(RR^T) + Tr(RB_{00}R^T)$ ← EXCESS NOISE (?)

$B_{00} \triangleq B \ni m = m_0 \;\&\; b = 0$

ADDITIVE MODEL

$E\|\hat{m} - m\|^2 = \sigma^2 Tr[RR^T]$

① LEAST SQUARES (LS) ESTIMATOR

(D, b) KNOWN

$R_{LS} = (D^T D)^{-1} D^T$

② MINIMUM VARIANCE UNBIASED EST.

$\underline{D} = \underline{b}$, $\underline{\overline{m}} = \underline{m_0}$

$R_{MVU} = (D^T B_0^{-1} D)^{-1} D^T B_0^{-1}$

B_0 = DIAGONAL W/ $\overline{\underline{n}} = D m_0 + \underline{b}$ ON DA DIAGONAL

③ MODIFIED MVU

KNOW D, b , $B_0 = \frac{1}{M} \sum_{i=1}^{M} (m_0)_i$

DIS IS, IN EFFECT, AN MVU WITH FAULTY INFO 'BOUT $\overline{\underline{m}} = m$

$R_{MMVU} =$

I. STATISTICAL COMMUNICATIONS REVIEW

- SOME DISTRIBUTIONS

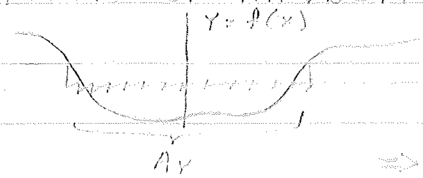
  - GAUSSIAN: $p_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$

  - RALEIGH: $\sqrt{G_1^2 + G_2^2} \Rightarrow p_X(x) = \frac{x}{b} e^{-x^2/2b} \mu(x)$

  - χ_n^2: $\Sigma G_i^2 \Rightarrow p_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-x/2} \mu(x)$

- TRANSFORMATION OF RANDOM VARIABLES

  $Y = f(x)$



  $\Rightarrow F_Y(Y) = P[Y \le Y]$

  $= P[X \in A_Y]$

  $\Rightarrow p_Y(Y) = \frac{d}{dY} P[X \in A_Y]$

  - EX $f(x) = aX^2$

  $F_Y(Y) = P[X \in A_Y]$

  $= P\left[-\sqrt{\frac{Y}{a}} \le X \le \sqrt{\frac{Y}{a}}\right]$

  $= \int_{-\infty}^{\sqrt{Y/a}} p_X(x)\,dx - \int_{-\infty}^{-\sqrt{Y/a}} p_X(x)\,dx$

  $\frac{d}{dY} F_Y(Y) = p_Y(Y) = \frac{1}{2\sqrt{aY}}\left[p_X\left(\sqrt{\frac{Y}{a}}\right) + p_X\left(-\sqrt{\frac{Y}{a}}\right)\right]$

  - EX $f(x)$ MONOTONIC : $Y = f(x)$; $X = f^{-1}(Y)$

  $\Rightarrow p_Y(Y) = p_X[f^{-1}(Y)] \left|\frac{dX}{dY}\right|$; $\left|\frac{dX}{dY}\right|$ = JACOBIAN

  - EX $Y = \cos X$; $p_X(x) = \frac{1}{\pi} \operatorname{rect}\left[\frac{X - \pi/2}{\pi}\right]$

  THIS TRANSFORMATION IS MONOTONIC FOR $0 < X < \pi$

  $\Rightarrow p_Y(Y) = \frac{1}{\pi}\left|\frac{dX}{dY}\right| = \frac{1}{\pi\sqrt{1-Y^2}} \operatorname{rect}\left[\frac{Y}{2}\right]$

- CONDITIONAL PROBABILITIES

  $P(X_k, Y_j) = P(X_k)\,P(Y_j/X_k) = P(Y_j)\,P(X_k/Y_j)$

- JOINT DISTRIBUTIONS

  $F_{XY}(x,Y) = P[X \le x \text{ AND } Y \le Y]$

  $p_{XY}(x,Y) = \frac{\delta^2}{\delta x\,\delta Y} F_{XY}(x;Y)$

  $= p_X(x)\,p_{Y/X}(Y/x) = p_Y(Y)\,p_{X/Y}(x/Y)$

- STATISTICAL INDEPENDENDENT

  IF $p_{Y/X}(Y/x) = p_Y(Y)$ OR $p_{X/Y}(x/Y) = p_X(x)$

  $p_{XY}(x,Y) = p_X(x)\,p_Y(Y)$

- MULTIVARIATE PROBABILITY TRANSFORMATIONS

GIVEN $p_{XY}(x,y)$ FIND $p_{UV}(u,v)$ GIVEN

$U = f(X,Y)$ AND $V = g(X,Y)$

$A_{uv} \triangleq \{ \text{ALL } (x,y) \ni U \le u \text{ AND } V \le v \}$

$p_{UV}(u,v) = \frac{\partial^2}{\partial u \partial v} P[(x,y) \in A_{uv}]$

FOR MONOTONIC $f(x,y)$

$$\left. \begin{array}{l} u = f(x,y) \\ v = g(x,y) \end{array} \right\} \left\{ \begin{array}{l} x = F(u,v) \\ y = G(u,v) \end{array} \right.$$

$$p_{UV}(u,v) = p_{XY}(x = F(u,v),\ y = G(u,v))\,|J|$$

$$|J| = \text{JACOBIAN DETERMINANT} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

NOTE: $dx\,dy = |J|\,du\,dv$; $|J| = 1 \Rightarrow$ "AREA CONSERVING" XFORM

- STATISTICAL AVERAGES

- $E[f(x)] = \overline{f(x)} = \int_{-\infty}^{\infty} f(x)\, p_X(x)\,dx$

MEAN $= \overline{X} = E(x)$

VARIANCE: $\overline{(x - \overline{x})^2} = \overline{x^2} - \overline{x}^2 = \sigma^2$

STANDARD DEVIATION: $\sigma = \sqrt{\sigma^2}$

- THEOREM: IF $Y = f(x)$, THEN $\overline{Y} = \int_{-\infty}^{\infty} y\, p_Y(y)\,dy = \int_{-\infty}^{\infty} f(x)\, p_X(x)\,dx$

- CHARACTERISTIC FUNCTION

$$M_X(j\omega) = \mathcal{F}^{-1}[p_X(x)] = E[e^{j\omega x}]$$
$$= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} \overline{X^n}$$

$$\overline{x^n} = (j)^{-n} \frac{d^n}{d\omega^n} M_X(j\omega) \Big|_{\omega = 0}$$

- EXAMPLES OF CHARACTERISTIC FUNCTIONS

- BINOMIAL: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-1}$ $\quad x = 0, 1, \ldots n$

$$M_X(j\omega) = (p e^{j\omega} + q)^n$$

- POISSON: $p_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}$; $M_X(j\omega) = e^{\lambda(e^{j\omega} - 1)}$
- UNIFORM: $p_X(x) = \frac{1}{b-a}$; $a < x < b$; $M_X(j\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$
- GAUSSIAN: $p_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$; $M_X(j\omega) = e^{j\omega m - \frac{1}{2}\omega^2\sigma^2}$

- TCHEBYCHEFF'S INEQUALITY

  $\forall\ p_X(x)$ WITH FINITE MEAN m & VARIANCE σ^2

  $$P[|x-m| \geq \alpha\sigma) \leq 1/\alpha^2 \qquad \forall\ \alpha > 0$$

- CORRELATION COEFFICIENT (BETWEEN X & Y)

  $$\rho \triangleq \frac{\overline{XY} - \bar{X}\bar{Y}}{\sigma_x \sigma_Y}$$

  $\rho = 1 \Rightarrow$ PERFECTLY CORRELATED, $\rho = 0 \Rightarrow$ UNCORRELATED

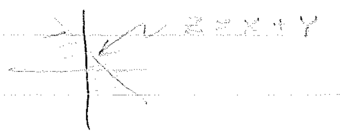
  TWO STATISTICALLY IND. R.V. ARE UNCORRELATED

  (CONVERSE NOT NECESSARILY TRUE EXCEPT FOR GAUSSIAN)

- COVARIANCE: $cov(X,Y) = \overline{XY} - \bar{X}\bar{Y} = \rho\sigma_x\sigma_Y$

- DISTRIBUTIONS OF COMBINED STATISTICS

  - GIVEN $p_{XY}(x,y)$ AND $Z = X+Y$, WHAT IS $p_Z(z)$?



$$F_Z(z) = P[Z \leq z] = \int_{-\infty}^{\infty} dy \int_{-\infty}^{z-y} p_{XY}(x,y)\,dx = \int_{-\infty}^{\infty} dy \int_{-\infty}^{z} d\xi\, p_{XY}(\xi - y; y)$$

$$p_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} p_{XY}(z-y; y)\,dy$$

  MAY ALSO APPY MULTIVARIATE PROBABILITY TRANSFORMS

  - CENTRAL LIMIT THEOREM

    $\{X_n\}$ ARE R.V.'S WITH MEANS $\{\mu_n\}$ AND VARIANCES $\{\sigma_n^2\}$

    THEN IF $\exists$ AN $m, M \ni \sigma_i^2 > m > 0$ AND $\overline{(X_i - \mu_i)^3} < M$

    $\forall\ i$, THEN IF $Z = \frac{1}{(\Sigma\sigma_i^2)^{1/2}} \sum_{i=1}^{n} (x_i - \mu_i) = \frac{1}{(\Sigma\sigma_i^2)^{1/2}} (\Sigma X_i - \Sigma\mu_i)$

    AS $n \to \infty$, $Z \sim \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

  - ADDITIONAL COMBINATIONS

$$p_{X-Y}(\xi) = \int_{-\infty}^{\infty} dx\, p_{XY}(\xi + x; x) = \int_{-\infty}^{\infty} dx\, p_{XY}(x; x - \xi)$$

$$p_{XY}(\xi) = \int_{-\infty}^{\infty} \frac{dx}{|x|} p_{XY}\left(\frac{\xi}{x}; x\right) = \int_{-\infty}^{\infty} dx \frac{1}{|x|} p_{XY}\left(x, \frac{\xi}{x}\right)$$

$$p_{X/Y}(\xi) = \int_{-\infty}^{\infty} dx\, |x|\, p_{XY}(\xi x, \xi)$$

- RANDOM PROCESSES
  - STATIONARITY (STRICTLY)
    - ORDER 1 $\Rightarrow p_X[x(t_1)] = p_X[x(t_1+h)] \quad \forall\, h, t_1$
    - ORDER 2 $\Rightarrow p_X[x(t_1), x(t_1-\tau)] = p_X[x(t_1+h), x(t_1-\tau+h)]$
    - WIDE SENSE STATIONARY:
      1. $E(X)$ INDEP. OF t 2. $E[x(t_1)x(t_2)]$ DEPENDS ON t_1-t_2

      ALL ORDER 2'S ARE WSS. (CONVERSE NOT NEC. TRUE)
  - ERGODICITY:

    SAMPLE AVERAGE: $\widetilde{X^n(t)} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X_T^{\,n}(t')\,dt'$

    ITS NECESSARY THAT $\widetilde{X^n(t)} = \overline{X^n(t)}$

    ALL ERGODIC PROCESSES ARE STATIONARY
  - AUTOCORRELATION & CROSSCORRELATION
    - $R_X(t_1, t_2) = \overline{X(t_1)X^*(t_2)} = \iint_{-\infty}^{\infty} x_1 x_2^*\, p_X(x_1, x_2; t_1, t_2)\,dx_1\,dx_2$

      FOR WIDE-SENSE STATIONARY: $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$

      FOR ERGODICITY: $R_X(\tau) = \mathcal{R}_X(\tau)$

      $$R_X(\tau) = R_X(-\tau) \; ; \quad R_X(0) = \sigma^2 \geq |R_X(\tau)|$$
    - $R_{XY}(t_1, t_2) \triangleq \overline{X^*(t_1)Y(t_2)}$

      FOR JOINTLY WSS $R_{XY}(t_1, t_2) = R_{XY}(\tau)$

      $$R_{XY}(-\tau) = R_{YX}{}^*(\tau)$$

      FOR JOINT ERGODICITY: $R_{XY}(\tau) = \mathcal{R}_X(\tau)$
    - IF $Z = X + Y$

      $$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$
    - $\operatorname{COV}[X(t)] = R_X(\tau) - \overline{X(t)}^{\,2}$
    - PASSING THROUGH TIME-INVARIANT NETWORK

      $$\xrightarrow{X} \boxed{h(t)} \xrightarrow{Y}$$

      $$R_Y(\tau) = C(\tau) * R_X(\tau)$$

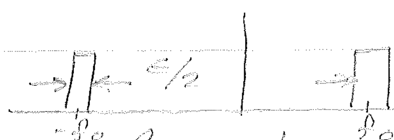
      $$C(\tau) = h(t) * h(t) = \int_{-\infty}^{\infty} h(t)\,h(t+\tau)\,dt$$

- POWER SPECTRAL DENSITY
  - $\phi_x(f) = \mathcal{F}[R_x(\tau)] = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f \tau} d\tau$

    $R_x(\tau) = \mathcal{F}^{-1}[\phi_x(f)]$
  - PASSING THROUGH A TIME-INVARIANT NETWORK

    $\phi_Y(f) = |H(f)|^2 \phi_Y(f)$
  - PROPERTIES
    - FOR ERGODICITY:

      $\int_{-\infty}^{\infty} \phi_x(f) df = R_x(0) = \overline{X^2} = P_{AVE} = \widetilde{X^2}$
    - PASSING THROUGH BANDPASS FILTER



$\overline{Y^2} \approx \frac{\epsilon}{2}\phi_x(f_0) + \frac{\epsilon}{2}\phi_x(f_0)$

  - $\phi(f) = \lim_{T \to \infty} \frac{1}{2T} E[|X_T(f)|]^2$
  - $\phi(f) = N_0/2 \Longrightarrow$ WHITE NOISE
  - SUMMARY
    - $\int_{-\infty}^{\infty} \phi(f) df = R_x(0) = \sigma_x^2$
    - $\phi(f)$ IS NON-NEGATIVE REAL & EVEN

II. NOTATION & FOURIER XFORM DEFINITIONS

- PHASORS & ANALYTIC SIGNALS

$$u(t) = A\cos(2\pi\nu_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j2\pi\nu_0 t} + \frac{A}{2} e^{-j\phi} e^{-j2\pi\nu_0 t}$$

$$\underline{u}(t) = A\, e^{j\phi} e^{j2\pi\nu_0 t} = \underline{A}\, e^{j2\pi\nu_0 t}$$

$$\underline{A} = A e^{j\phi} \; ; \; Re[\underline{u}(t)] = u(t)$$

$$U(\nu) = \mathcal{F}[u(t)] = \frac{A}{2} e^{j\phi}\delta(\nu - \nu_0) + \frac{A}{2} e^{-j\phi}\delta(\nu + \nu_0)$$

$$\underline{U}(\nu) = \mathcal{F}[\underline{u}(t)] = A e^{j\phi}\delta(\nu - \nu_0)$$

GOING FROM $u(t)$ TO $\underline{u}(t)$, WE DOUBLE POSITIVE FREQUENCIES & SUPPRESS NEGATIVE FREQUENCIES

- GENERALIZING: THE ANALYTIC SIGNAL

$$\underline{u}(t) = 2\int_0^\infty U(\nu)\, e^{j2\pi\nu t}\, dt = \text{ANALYTIC SIGNAL}$$

(1) $Re[\underline{u}(t)] = u(t)$

PROOF: $Re[\underline{u}(t)] = \frac{1}{2}[\underline{u}(t) + \underline{u}^*(t)]$

$$= \int_0^\infty U(\nu)\, e^{j2\pi\nu t} d\nu + \int_0^\infty U^*(\nu)\, e^{-j2\pi\nu t} d\nu$$

WHERE $U(\nu) = \mathcal{F}[u(t)]$

$u(t)$ IS REAL $\Rightarrow U(\nu) = U^*(-\nu)$

$$\Rightarrow Re[\underline{u}(t)] = \int_0^\infty U(\nu)\, e^{j2\pi\nu t} d\nu + \int_0^\infty U(-\nu)\, e^{-j2\pi\nu t} d\nu$$

$$= \int_{-\infty}^\infty U(\nu)\, e^{j2\pi\nu t} dt = u(t)$$

(2) $Im[\underline{u}(t)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t')\,dt'}{t - t'}$ (HILBERT XFORM)

PROOF: $Im[\underline{u}(t)] = \frac{1}{j2}[\underline{u}(t) - \underline{u}^*(t)]$

$$= \frac{1}{j}\int_0^\infty U(\nu)\, e^{j2\pi\nu t} - \frac{1}{j}\int_0^\infty U^*(\nu)\, e^{-j2\pi\nu t} d\nu$$

$$= -j\int_0^\infty U(\nu) e^{j2\pi\nu t} + \int_{-\infty}^0 (+j)\, U(\nu)\, e^{j2\pi\nu t} d\nu$$

$$\Rightarrow \mathcal{F}[Im\, \underline{u}(t)] = -j\, \mathrm{sgn}\, \nu \cdot \mathcal{F}[u(t)]$$

$$Im\, \underline{u}(t) = \mathcal{F}^{-1}[-j\, \mathrm{sgn}\, \nu] * u(t)$$

$$= \frac{1}{\pi t} * u(t)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t')}{t - t'} dt$$

SIMILARLY, $u(t) = -\frac{1}{\pi}\int \frac{Im[\underline{u}(t')]}{t - t'} dt'$

$\Rightarrow u(t)$ AND $Im[\underline{u}(t)]$ ARE HILBERT XFORM PAIRS

→ SUMMARIZE

1) $\underline{u}(t)$ HAS ONLY POSITIVE FREQUENCIES

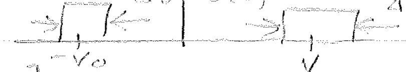
2) $Re[\underline{u}(t)] = u(t)$

3) $Im[\underline{u}(t)] = \mathcal{H}[u(t)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t')}{t'-t}\, dt'$

$$u(t) = \mathcal{H}^{-1}[Im\, \underline{u}(t)] = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Im[\underline{u}(t')]}{t'-t}\, dt'$$

4) $\mathcal{F}[Im\, \underline{u}] = -j\, \mathrm{sgn}\, \nu\, U(\nu) = -j\, \mathrm{sgn}\, \nu\, \mathcal{F}[u(t)]$

• COMPLEX ENVELOPES ON TIME VARYING PHASORS

NARROW BAND SPECTRUMS 

$$u(t) = A(t)\cos[2\pi\nu_0 t + \phi(t)]$$

$$\underline{u}(t) = A(t)\, e^{j\phi(t)}\, e^{j2\pi\nu_0 t}$$

$$\underline{A}(t) = A(t)\, e^{j\phi(t)} = \underline{u}(t)\, e^{-j2\pi\nu_0 t}$$

$$\underbrace{\int_{-\infty}^{\infty} u^2(t)\, dt = \infty}_{\text{ENERGY}} \; ; \; \underbrace{\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} u^2(t)\, dt}_{\text{AVERAGE PWR}} < \infty$$

- TIME AUTO-CORRELATION FUNCTION

$$\Gamma_{rr}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} u(t+\tau)\, u(t)\, dt$$

$$= \langle u(t+\tau)\, u(t)\rangle$$

1) $\Gamma_{rr}(0) = \langle u^2(t)\rangle$

2) $\Gamma_{rr}(\tau) = \Gamma_{rr}(-\tau)$

3) $|\Gamma_{rr}(\tau)| \le |\Gamma_{rr}(0)|$ ← FROM SCHWARZ:

$$\left[\int_{-T/2}^{T/2} u(t+\tau)\, u(t)\, dt\right]^2 \le \int_{-T/2}^{T/2} u^2(t+\tau)\, dt \int_{-T/2}^{T/2} u^2(t)\, dt$$

- POWER SPECTRAL DENSITY (WEINER-KHINTCHINE THEOREM)

$$G(\nu) = \mathcal{F}[\Gamma_{rr}(\tau)] = \int_{-\infty}^{\infty} \Gamma_{rr}\, e^{-j2\pi\nu\tau}\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{T}\, |U_T(\nu)|^2$$

$$\Rightarrow U_T(\nu) = \mathcal{F}[u(t)\, \mathrm{rect}(t/T)]$$

- ENSEMBLE AVERAGES FOR REAL SIGNALS

$$\Gamma_{rr}(t_2, t_1) = \overline{u(t_2)\, u(t_1)}$$

$$= \int_{-\infty}^{\infty}\!\!\int u_2\, u_1\, p(u_2, u_1)\, du_2\, du_1$$

$$\Gamma_{rr}(t_2, t_1) = \Gamma_{rr}(\tau) \Leftarrow \text{WIDE SENSE STATIONARY}$$

$$\Gamma_{rr}(\tau) = \Gamma_{rr}(\tau) \Leftarrow \text{ERGODIC}$$

- AUTOCORRELATION OF ANALYTIC SIGNALS

$$\underline{u}(t) = u(t) + j\,\mathcal{H}\{u(t)\} = u(t) + j\,u^i(t)$$

$$\underline{\Gamma}(\tau) \triangleq \langle \underline{u}(t+\tau)\, \underline{u}^*(t) \rangle$$

1) $\underline{\Gamma}(0) = \langle |\underline{u}(t)|^2 \rangle = \langle |u(t)|^2 \rangle + \langle |u_i(t)|^2 \rangle$

2) $\underline{\Gamma}(\tau) = \underline{\Gamma}^*(-\tau) \Rightarrow$ HERMITIAN

3) $|\underline{\Gamma}(\tau)| \leq |\underline{\Gamma}(0)|$

$$\underline{\Gamma}(\tau) = [\underbrace{\Gamma_{rr}(\tau) + \Gamma_{ii}(\tau)}_{\text{AUTO CORRELATIONS}}] + j\,[\underbrace{\Gamma_{ir}(\tau) - \Gamma_{ri}(\tau)}_{\text{CROSS-CORRELATIONS}}]$$

$$\begin{cases} \mathcal{F}[\Gamma_{rr}(\tau)] = \mathcal{F}[\Gamma_{ii}(\tau)] = G(\nu) \\ \mathcal{F}[\Gamma_{ri}(\tau)] = -\mathcal{F}[\Gamma_{ir}(\tau)] = j\,\mathrm{sgn}\,\nu\; G(\nu) \end{cases}$$

THUS, IN SUMMARY

1) $\Gamma_{rr}(\tau) = \Gamma_{ii}(\tau)$

2) $\Gamma_{ri}(\tau) = -\Gamma_{ir}(\tau)$

3) $\underline{\Gamma}(\tau) = 2\Gamma_{rr}(\tau) + j\,2\Gamma_{ir}(\tau) = \Gamma^r(\tau) + j\,\Gamma^i(\tau)$

4) $\mathcal{F}[\underline{\Gamma}(\tau)] = 4\, G(\nu)\, \mu(\nu)$

5) $\underline{\Gamma}(\tau)$ IS ANALYTIC

$$\Rightarrow \Gamma^i(\tau) = \mathcal{H}[\Gamma^r(\tau)]$$

- CROSS CORRELATION FUNCTIONS OF ANALYTIC SIGNALS

$\Gamma_{12}(\tau) = \langle u_1(t+\tau)\, u_2^*(\tau) \rangle$

1) $\Gamma_{12}(0) = \langle u_1(t)\, u_2(t) \rangle$

2) $\Gamma_{12}(\tau) = \Gamma_{21}^*(-\tau)$

3) $|\Gamma_{12}(\tau)| \leq |\Gamma_{11}(0)\, \Gamma_{22}(0)|^{1/2}$

$\Gamma_{11}(\tau) = \langle u_1(t+\tau)\, u_1^*(t) \rangle$

$\Gamma_{22}(\tau) = \langle u_2(t+\tau)\, u_2^*(t) \rangle$

FOR NARROW-BAND SIGNALS:

$u_1(t) = A_1(t)\, e^{j2\pi\nu_0 t}$; $u_2(t) = A_2(t)\, e^{j2\pi\nu_0 t}$

$\Gamma_{12}(\tau) = \langle A_1(t+\tau)\, A_2^*(t) \rangle\, e^{j2\pi\nu_0\tau}$

- MUTUAL OR CROSS SPECTRAL DENSITY

$G_{12}(\nu) = \mathcal{F}\{\langle u_1(t+\tau)\, u_2(\tau) \rangle\}$

$\Gamma_{12}(\tau) = 4\int_0^\infty G_{12}(\nu)\, e^{j2\pi\nu\tau}\, d\nu$

Problem Set #2

EE 349
JWG

Due Tuesday
April 15

#1 Show that the characteristic function of the intensity of polarized thermal light is given by

$$M_I(\omega) = \frac{1}{1 - j\omega\bar{I}}$$

#2. Show that the standard deviation σ_I of the intensity of partially polarized thermal light is

$$\sigma_I = \sqrt{\frac{1+P^2}{2}}\;\bar{I}$$

#3 Right-hand circularly polarized light is described by orthogonal polarization components

$$u_x(t) = A(t)\, e^{j2\pi\bar{\nu}t}$$

$$u_y(t) = A(t)\, e^{j(2\pi\bar{\nu}t + \frac{\pi}{2})}$$

where $\underline{A}(t)$ is a slowly varying complex envelope. Show that such light is not unpolarized by our definition of unpolarized light.

P4. Consider a single mode laser emitting light described by the analytic signal

$$\underline{u}(t) = \exp\{-j[2\pi\nu_0 t - \theta(t)]\}$$

(a) Show that the autocorrelation function of $u(t)$ is given by

$$\underline{\Gamma}_U(t_2,t_1) = e^{-j2\pi\nu_0\tau}\, \underline{M}_{\Delta\theta}(1)$$

where $\underline{M}_{\Delta\theta}(\omega)$ is the characteristic function of the phase difference $\Delta\theta = \theta(t_2) - \theta(t_1)$.

$e^{-\frac{1}{2}\sigma^2 ...}$

(b) Show that for a gaussian $\theta(t)$ arising from a stationary instantaneous phase frequency model [illegible]

$$\underline{\Gamma}_U(\tau) = e^{j2\pi\nu_0\tau}\, e^{-\frac{1}{2}D_\theta(\tau)}$$

where $D_\theta(\tau)$ is the structure function of the phase process $\theta(t)$.

#5. ~~Fraunhofer~~ Beginning with the equation at the bottom of page 7, show that if $\Delta\nu \ll \bar{\nu}$ and $r \ll c/\Delta\nu$ for all P, then

$$\underline{u}(P_0,t) \cong \iint_\Sigma \frac{e^{j2\pi\frac{r}{\bar\lambda}}}{j\bar\lambda r}\, \underline{u}(P_1,t)\, dS$$

can be used to describe the propagation of $\underline{u}(P,t)$.

#6. Consider the wave emitted by a laser oscillating in N equal strength, independent modes.

(a) Show that the total intensity I_T can be expressed in terms of the intensity I_1 of one mode and the intensity I_{0} [illegible] from the remaining $N-2$ modes by

$$I_N = I_{N-1} + I_1 + 2\sqrt{I_{N-1} I_1}\cos\phi$$

where ϕ is uniformly distributed on $(-\pi, \pi)$, and independent of I_{N-1}.

(b) Show that

$$\bar{\bar{I}}_{N-1} = \frac{N-1}{N}\bar{\bar{I}}_N$$

(c) Prove that the standard deviation σ_N of I_N satisfies

$$\frac{\sigma_N}{\bar{\bar{I}}} = \left[\left(\frac{N-1}{N}\right)^2\left(\frac{\sigma_{N-1}}{\bar{\bar{I}}}\right)^2 + \left(\frac{N-1}{N}\right)^2 + \frac{1}{N^2} + \frac{4(N-1)}{N^2} - 1\right]^{1/2}$$

where $\bar{\bar{I}} = \bar{\bar{I}}_N$.

PROBLEMS

1 - Let the random process $U(t)$ be defined by

$$U(t) = A \cos (2\pi W t - \Phi)$$

where W is a known constant, Φ is uniformly distributed on $(-\pi, \pi)$, and the probability density function of A is given by

$$p_A(a) = \frac{1}{2}\delta(a-1) + \frac{1}{2}\delta(a-2).$$

(a) Calculate $\langle u^2(t)\rangle$ for a sample function with amplitude 1 and a sample function with amplitude 2.

(b) Calculate $\overline{u^2}$.

(c) Show that

$$\overline{u^2} = \frac{1}{2}\langle u^2\rangle_1 + \frac{1}{2}\langle u^2\rangle_2$$

where $\langle u^2\rangle_1$ and $\langle u^2\rangle_2$ represent the results of part (a) for amplitudes of 1 and 2, respectively.

2 - Consider the random process $U(t)=at$, where a is a random variable uniformly distributed on $(-1, 1)$.

(a) Sketch some sample functions of this process.

(b) Find the time —— autocorrelation function of $U(t)$.

OP 8 - Assuming U(t) is wide-sense stationary, with mean $\bar{u}$ and variance σ^2, which of the following functions represent possible structure functions for U(t) ?

(a) $D_U(\tau) = 2\sigma^2 \left[1 - e^{-\alpha|\tau|}\right]$

(b) $D_U(\tau) = 2\sigma^2 \left[1 - e^{-\alpha|\tau|} \cos \omega\tau\right]$

(c) $D_U(\tau) = 2\sigma^2 \left[1 - \sin \omega\tau\right]$

(d) $D_U(\tau) = 2\sigma^2 \left[1 - \cos \omega\tau\right]$

(e) $D_U(\tau) = 2\sigma^2 \left[1 - \mathrm{rect}\, \omega\tau\right]$

9 - Prove that the Hilbert transform of the Hilbert transform of u(t) is −u(t), up to a possible additive constant.

OP 10 - Parseval's theorem, in generalized form, states that for any two Fourier transformable functions $\mathbf{f}(t)$ and $\mathbf{g}(t)$ with transforms $\mathbf{F}(\nu)$ and $\mathbf{G}(\nu)$,

$$\int_{-\infty}^{\infty} \mathbf{f}(t)\mathbf{g}^*(t)\,dt = \int_{-\infty}^{\infty} \mathbf{F}(\nu)\mathbf{G}^*(\nu)\,d\nu \,.$$

Show that, if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are analytic signals,

$$\int_{-\infty}^{\infty} \mathbf{u}(t)\, \mathbf{v}(t)\, dt = 0 \,.$$

a) Find the probability density function of the random variable Z defined by

$$Z = \Theta_1 + \Theta_2$$

b) If Z represents a phase angle that can only be measured modulo 2π, show that, despite the result of a), Z is uniformly distributed on $(-\pi, \pi)$.

OP (10) Consider the random phasor sum of section 2-1 with the single change that the phases ϕ_k are uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Find the following quantities : $\bar{r}$, $\bar{i}$, σ_r^2, σ_i^2 and ρ_{ri}. Make a rough plot of the contours of constant probability in the complex plane.

OP (11) Let the random variables U_1 and U_2 be jointly gaussian, with zero means, equal variances and correlation coefficient $\rho \neq 0$. Consider new random variables V_1 and V_2 defined by a rotational transformation about the origin of the (U_1, U_2) plane,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where ϕ is the rotation angle. Show that, if ϕ is chosen to be $45°$, V_1 and V_2 are <u>independent</u> random variables. What are the means and variances of V_1 and V_2 in this case?

(12) Consider n independent random variables $W_1, W_2, \ldots W_n$, each of which obeys a Cauchy density function,

$$p_W(w) = \frac{1}{\pi\beta\left[1 + \left(\frac{w}{\beta}\right)^2\right]}.$$

a) Show that this density function violates one of the conditions (6-16) associated with the validity of the central limit theorem.

b) Show that the random variable

$$Y = \frac{1}{n}\sum_{i=1}^{n} W_i$$

obeys a Cauchy distribution for all n.

OP. (13) A certain computer contains a random number generator which generates numbers with uniform relative frequencies (or probability density) on the interval (0,1). Suppose, however, that it is desired to simulate trials of a random variable Z with density function $p_Z(z)$ that is not uniform.

a) If the values generated by the computer are represented by u, with

$$p_U(u) = \begin{cases} 1 & 0 < u \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

show that, by means of a monotonic transformation $z = g(u)$, it is possible to obtain the desired $p_Z(z)$, and that, if $u = G(z)$ represents the inverse of $g(\cdot)$, then G should be chosen to satisfy

$$G(z) = \pm \int p_Z(z)\, dz$$

where $\int$ is an indefinite integral and the + or - sign

① For polarized thermal light

$$p_I(I) = \begin{cases} \frac{1}{\bar{I}} e^{-I/\bar{I}} , & I \geq 0 \\ 0 , & I < 0 \end{cases}$$

$$\therefore \; M_I(\omega) \triangleq E\left[e^{j\omega I}\right] = \int_0^\infty e^{j\omega I}\left[\frac{1}{\bar{I}} e^{-I/\bar{I}}\right] dI$$

$$= \frac{1}{\bar{I}} \left\{ \frac{\exp\left[I\left(j\omega - \frac{1}{\bar{I}}\right)\right]}{j\omega - \frac{1}{\bar{I}}} \right\} \Bigg|_0^\infty$$

$$\boxed{M_I(\omega) = \frac{1}{1 - j\omega\bar{I}}}$$

② For partially polarized light (see notes)

$$M_I(\omega) = \left[\frac{1}{1 - j\frac{\omega}{2}(1+P)\bar{I}}\right]\left[\frac{1}{1 - j\frac{\omega}{2}(1-P)\bar{I}}\right]$$

$$= \frac{(1+P)/2P}{1 - j\frac{\omega}{2}(1+P)\bar{I}} - \frac{(1-P)/2P}{1 - j\frac{\omega}{2}(1-P)\bar{I}}$$

Now we evaluate the moments of interest as follows: $\longrightarrow$

$$\frac{\partial M_I(\omega)}{\partial \omega} = \frac{1+\rho}{2\rho} \frac{\frac{j}{2}(1+\rho)\bar{I}}{\left[1-j\frac{\omega}{2}(1+\rho)\bar{I}\right]^2}$$

$$- \frac{1-\rho}{2\rho} \frac{\frac{j}{2}(1-\rho)\bar{I}}{\left[1-j\frac{\omega}{2}(1-\rho)\bar{I}\right]^2}$$

$$\therefore \quad \left.\frac{\partial M_I(\omega)}{\partial \omega}\right|_{\omega=0} \triangleq E\left[j I e^{j\omega I}\right]\Big|_{\omega=0}$$

$$= j E[I] = j\bar{I}$$

Now take a second partial derivative to get the result

$$\frac{\partial^2 M_I(\omega)}{\partial \omega^2} = \frac{1+\rho}{2\rho} \frac{(-\frac{1}{2})(1+\rho)^2 \bar{I}^2}{\left[1-j\frac{\omega}{2}(1+\rho)\bar{I}\right]^3}$$

$$- \frac{1-\rho}{2\rho} \frac{(-\frac{1}{2})(1-\rho)^2 \bar{I}^2}{\left[1-j\frac{\omega}{2}(1-\rho)\bar{I}\right]^3}$$

$$\Rightarrow \overline{I^2} = \left.\frac{-\partial^2 M_I(\omega)}{\partial \omega^2}\right|_{\omega=0} = \frac{1}{4\rho}\left[(1+\rho)^3-(1-\rho)^3\right]\bar{I}^2$$

$$= \frac{1}{2}(3+\rho^2)\bar{I}^2$$

$$\sigma_I^2 = [illegible] - [illegible] = \left[\tfrac{1}{2}(3+\sigma^2) - 1\right]\bar{I}^2$$

$$= \frac{1}{2}(1+\sigma^2)\bar{I}^2$$

$$\Rightarrow \boxed{\sigma_I = \sqrt{\frac{1+\sigma^2}{2}}\;\bar{I}}$$

③

$$\underline{u}_x(t) = \underline{A}(t)\, e^{-j2\pi\bar{\nu}t}$$

$$\underline{u}_y(t) = \underline{A}(t)\, e^{-j(2\pi\bar{\nu}t + \frac{\pi}{2})}$$

From the notes, the second requirement for unpolarized light is

$$\langle \underline{u}_x^*(t+\tau)\, \underline{u}_y(t) \rangle = 0 \quad \text{for all } \tau$$

What if $\tau = 0$? Then we have

$$\langle \underline{u}_x^*(t)\, \underline{u}_y(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \underline{A}^*(t)\, \underline{A}(t)\, e^{-j\frac{\pi}{2}}\, dt$$

$$= -j\, \langle |\underline{A}(t)|^2 \rangle \neq 0 \text{ in general}$$

hence, unless $A(t) \equiv 0$ for all t, it is not unpolarized.

(a) $\Gamma_u(t_2, t_1) = \overline{\underline{u}(t_2)\, \underline{u}^*(t_1)}$

$$= E\left[e^{-j2\pi\nu_0(t_2-t_1)} e^{j[\theta(t_2)-\theta(t_1)]}\right]$$

$$= e^{-j2\pi\nu_0\tau} E\left[e^{j\Delta\theta}\right] = e^{-j2\pi\nu_0\tau} E\left[e^{j\omega\Delta\theta}\right]_{\omega=1}$$

$$= e^{-j2\pi\nu_0\tau} M_{\Delta\theta}(1)$$

(b) Since $\Delta\theta = \theta(t_2) - \theta(t_1)$

and $\theta(t_2), \theta(t_1)$ are Gaussian ~~process~~ random variables, thus a linear combination of them is also Gaussian. Forthermore

$$\overline{\Delta\theta} = \overline{\theta(t_2)} - \overline{\theta(t_1)} = 0$$

$$\sigma^2_{\Delta\theta} = \overline{(\Delta\theta)^2} - \overline{\Delta\theta}^2 = \overline{\Delta\theta^2} \equiv D_\theta(\tau)$$

Now we recall that for a Gaussian process ($\Delta\theta$ is GRP).

$$M_{\Delta\theta}(\omega) = \exp\left[j\omega\cancelto{0}{\overline{\Delta\theta}} - \frac{\omega^2\sigma^2_{\Delta\theta}}{2}\right]$$

$$\Rightarrow M_{\Delta\theta}(1) = \exp\left[-\tfrac{1}{2} D_\theta(\tau)\right]$$

$$\Rightarrow \Gamma_u(\tau) = e^{-j2\pi\nu_0\tau} e^{-\frac{1}{2} D_\theta(\tau)}$$

(c) For [illegible], then $K(\theta) \approx 1$ (obliquity factor ≈ 1) and we have.

$$\underline{u}_T(P_0,t) = \iint_{\Sigma} \frac{2}{jcr} \int_0^{\infty} \nu \underline{U}_T(P_1,\nu) e^{-j2\pi\nu(t-\frac{r}{c})} d\nu\, ds$$

Since $\Delta\nu \ll \bar{\nu}$, $\Rightarrow \nu \cong \bar{\nu}$

Now use the following two approximations

① $\dfrac{\bar{\nu}}{c} = \bar{\lambda}$, $\Rightarrow \dfrac{\nu}{c} \approx \bar{\lambda}$

② $e^{-j2\pi\nu\frac{r}{c}} \cong e^{-j2\pi\bar{\nu}\frac{r}{c}}$

$$\therefore \nu \cdot \frac{r}{c} \ll \frac{\nu}{\Delta\nu} \approx \frac{\bar{\nu}}{\Delta\nu} \gg 1$$

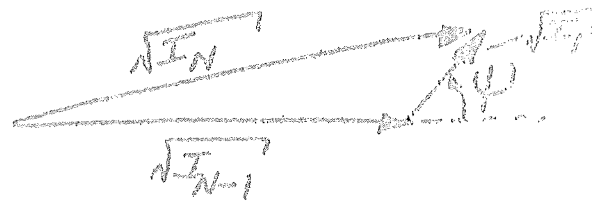
we can now write

$$\underline{u}_T(P_0,t) \approx \iint_{\Sigma} \frac{e^{j2\pi\bar{\nu}\frac{r}{c}}}{j\bar{\lambda}r} \left[\int_0^{\infty} 2U_T(P_1,\nu) e^{j2\pi\nu t} d\nu \right] ds$$

$$= \iint_{\Sigma} \frac{e^{j2\pi\frac{r}{\bar{\lambda}}}}{j\bar{\lambda}r} u_T(P_1,t)\, ds$$

and letting $T \to \infty$ gives the desired result.

(a) $\left(\sqrt{I_N}\right)^2 = \left(\sqrt{I_{N-1}}\right)^2 + \left(\sqrt{I_1}\right)^2 + 2\sqrt{I_{N-1} I_1}\cos\psi$



(b) $\overline{I_N} = \overline{I_{N-1}} + \overline{I_1} + 2\,\overline{\sqrt{I_{N-1} I_1}}\;\cancelto{0}{\overline{\cos\psi}}$

$= \overline{I_{N-1}} + \overline{I_1}$

$$\left.\begin{aligned}
\text{i.e.}\quad \overline{I_2} &= \overline{I_1} + \overline{I_1} = 2\overline{I_1}\\
\overline{I_3} &= \overline{I_2} + \overline{I_1} = 3\overline{I_1}\\
&\;\;\vdots\\
\overline{I_{N-1}} &= (N-1)\overline{I_1}\\
\overline{I_N} &= N\overline{I_1}
\end{aligned}\right\} \Rightarrow \overline{I_{N-1}} = \frac{N-1}{N}\overline{I_N}$$

(c) Since I_1 is a constant we have

$$\overline{I_N^2} = \overline{I_{N-1}^2} + I_1^2 + 2I_1\overline{I_{N-1}} + 4I_1\overline{I_{N-1}}\,\overline{\cos^2\psi}$$
$$+ 4\,\overline{I_{N-1}\sqrt{I_{N-1}I_1}}\;\cancelto{0}{\overline{\cos\psi}} + 4I_1\overline{\sqrt{I_{N-1}I_1}}\;\cancelto{0}{\overline{\cos\psi}}$$

$$\overline{\cos^2\psi} = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\psi\, d\psi = \frac{1}{2}, \quad \overline{I_{N-1}} = (N-1)I_1$$

$$\overline{I_N^2} = \overline{I_{N-1}^2} + 4(N-1)I_1^2 + I_1^2$$

$$\sigma_N^2 = \overline{I_N^2} - \overline{I_N}^2 = \overline{I_{N-1}^2} - (N^2 I_1^2) + 4(N-1)I_1^2 + I_1^2$$

$$= \overline{I_{N-1}^2} - (N-1)^2 I_1^2 + (N-1)^2 I_1^2 + 4(N-1)I_1^2 + I_1^2 - N^2 I_1^2$$

$$= \sigma_{N-1}^2 + (N-1)^2 I_1^2 + 4(N-1)I_1^2 + I_1^2 - N^2 I_1^2$$

$$\Rightarrow \frac{\sigma_N^2}{\overline{I_N}^2} = \left(\frac{N-1}{N}\right)^2\left(\frac{\sigma_{N-1}}{\overline{I_{N-1}}}\right)^2 + \frac{(N-1)^2}{N^2} + \frac{4(N-1)}{N^2} + \frac{1}{N^2} - 1$$

$$\Rightarrow \frac{\sigma_N}{\overline{I_N}} = \left[\left(\frac{N-1}{N}\right)^2\left(\frac{\sigma_{N-1}}{\overline{I_{N-1}}}\right)^2 + \left(\frac{N-1}{N}\right)^2 + \frac{1}{N^2} + \frac{4(N-1)}{N^2}\right]$$

BOB MARKS
STAT OPTICS
EE 5358
DUE 2/9/76

1. ASSUMING CIRCULAR COMPLEX GAUSSIAN RANDOM PROCESS:

10

$$p_I(I) = \frac{1}{\bar{I}} e^{-I/\bar{I}} \mu(I)$$

THUS

$$
\begin{aligned}
M_I(\omega) &= E\left[e^{j\omega I}\right] \\
&= \frac{1}{\bar{I}} \int_0^\infty e^{-I/\bar{I}} e^{j\omega I} \, dI \\
&= \frac{1}{\bar{I}} \int_0^\infty e^{-\left(\frac{1}{\bar{I}} - j\omega\right) I} \, dI \\
&= \frac{1}{\bar{I}} \frac{-1}{1/\bar{I} - j\omega} e^{-\left(\frac{1}{\bar{I}} - j\omega\right) I} \Big|_{I=0}^{\infty} \\
&= \frac{-1}{1 - j\omega \bar{I}} \left[0 - 1\right] \\
&= \frac{1}{1 - j\omega \bar{I}}
\end{aligned}
$$

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2. <u>ASSUMING</u> QUESTION MEANS PARTIALLY POLARIZED LIGHT:

$$\begin{cases} M_I(\omega) = \dfrac{1+\rho}{2\rho}\left[1 - j\dfrac{(1+\rho)\bar{I}}{2}\omega\right]^{-1} - \dfrac{1-\rho}{2\rho}\left[1 - j\dfrac{(1-\rho)\bar{I}}{2}\omega\right]^{-1} \\ \sigma_I^2 = \overline{x^2} - \bar{x}^2 \end{cases}$$

$$\overline{x^n} = (j)^{-n}\left(\frac{d}{d\omega}\right)^n M_x(\omega)\Big|_{\omega=0}$$

$$\bar{x} = (j)^{-1}\frac{d}{d\omega}M_I(\omega)\Big|_{\omega=0}$$

$$= \frac{1}{j}\left[\frac{1+\rho}{2\rho}\,\frac{j(1+\rho)\bar{I}}{2} \times \left\{1 - \frac{j(1+\rho)\bar{I}}{2}\omega\right\}^{-2} - \frac{(1-\rho)}{2\rho}\,\frac{j(1-\rho)\bar{I}}{2}\left\{1 - \frac{j(1-\rho)\bar{I}}{2}\omega\right\}^{-2}\right]_{\omega=0}$$

$$= \frac{(1+\rho)^2}{4\rho}\bar{I} - \frac{(1-\rho)^2}{4\rho}\bar{I}$$

$$= \frac{\bar{I}}{4\rho}\left[(1+\rho)^2 - (1-\rho)^2\right]$$

$$= \bar{I} \quad \text{(HOW 'BOUT THAT!)}$$

$$\overline{x^2} = j^{-2}\left[\frac{j(1+\rho)^2\bar{I}}{4\rho}(-2)\left(-\frac{j(1+\rho)\bar{I}}{2}\right)\left\{1 - \frac{j(1+\rho)\bar{I}}{2}\omega\right\}^{-3} - \frac{j(1-\rho)^2\bar{I}}{4\rho}(-2)\left(-\frac{j(1-\rho)\bar{I}}{2}\right)\left\{1 - \frac{j(1-\rho)\bar{I}}{2}\omega\right\}^{-3}\right]_{\omega=0}$$

$$= +\left[\left\{+\frac{(1+\rho)^3\bar{I}^2}{4\rho}\right\} - \left\{+\frac{(1-\rho)^3\bar{I}^2}{4\rho}\right\}\right]$$

$$= \frac{\bar{I}^2}{4\rho}\left[(1+\rho)^3 - (1-\rho)^3\right]$$

$$= \frac{\bar{I}^2}{4\rho}\left[(1+3\rho+3\rho^2+\rho^3) - (1-3\rho+3\rho^2-\rho^3)\right]$$

$$= \frac{\bar{I}^2}{4\rho}\left[6\rho + 2\rho^3\right]$$

$$= \frac{1}{4}\bar{I}^2(6+2\rho^2) = \frac{1}{2}\bar{I}^2(3+\rho^2)$$

$$\sigma^2 = \overline{x^2} - \bar{x}^2$$

$$= \left[\frac{1}{2}(3+\rho^2) - 1\right]\bar{I}^2 \qquad 3/2 - 2/2 = 1/2$$

$$= \frac{1}{2}(\rho^2+1)\bar{I}^2$$

$$\therefore\ \sigma = \sqrt{\sigma^2} = \sqrt{\frac{\rho^2+1}{2}}\,\bar{I}$$

3. $\underline{U}_x(t) = \underline{A}(t)\, e^{-j 2\pi \bar{\nu} t}$

10

$\underline{U}_Y(t) = \underline{A}(t)\, e^{-j(2\pi \bar{\nu} t + \pi/2)}$

CONDITIONS FOR UNPOLARIZED LIGHT:

1) LIGHT PASSED THROUGH POLARIZER IS INDEP. OF ROTATION

2) ANY TWO ORTHOGONAL COMPONENTS MUST HAVE PROPERTY:

$$\langle \underline{U}_x^*(t+\tau)\, U_Y(t) \rangle = 0$$

NOW

$$\langle U_x^*(t+\tau)\, U_Y(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \underline{U}_x^*(t+\tau)\, U_Y(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \underline{A}^*(t+\tau)\, \underline{A}(t)\, e^{j 2\pi \bar{\nu}(t+\tau)} \times e^{-j(2\pi \bar{\nu} t + \pi/2)}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \underline{A}^*(t+\tau)\, \underline{A}(t)\, e^{j 2\pi \bar{\nu}(\tau - \pi/2)}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} e^{j 2\pi \bar{\nu}(\tau - \pi/2)} \int_{-T/2}^{T/2} \underline{A}^*(t+\tau)\, \underline{A}(t)\, dt$$

$$= e^{j 2\pi \bar{\nu}(\tau - \pi/2)} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \underline{A}^*(t+\tau)\, \underline{A}(t)\, dt$$

$$= \langle \underline{A}^*(t+\tau)\, \underline{A}(t) \rangle\, e^{j 2\pi \bar{\nu}(\tau - \pi/2)}$$

$$= \underline{\Gamma}_A^*(t+\tau, t)\, e^{j 2\pi \bar{\nu}(\tau - \pi/2)}$$

WHERE $\underline{\Gamma}_A(t_1, t_2)$ IS THE AUTOCORRELATION OF THE COMPLEX RANDOM PROCESS $\underline{A}(t)$.

ONLY IF $\underline{\Gamma}_A(t_1, t_2)$ IS IDENTICALLY ZERO $\forall$ t_1 AND t_2 WILL THE LIGHT BE UNPOLARIZED. HOWEVER,

$$\underline{\Gamma}_A(t+\tau, t)\Big|_{\tau=0} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |A(t)|^2\, dt > 0 \Leftarrow$$

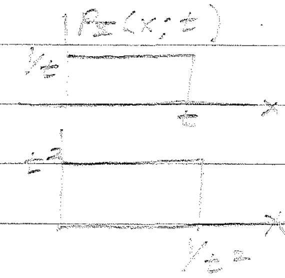
UNLESS $\underline{A}(t)$ IS IDENTICALLY ZERO (BORING ENVELOPE). THUS, THE LIGHT IS NOT UNPOLARIZED. ✓

(CONT. ⟶)

ACTUALLY, THIS ANALYSIS IS INCORRECT, SINCE THERE DO EXIST RANDOM PROCESSES FOR WHICH $\langle f(t) \rangle = 0$ WHERE $f(t) = |A(t)|^2 \geq 0$.

FOR EXAMPLE, CONSIDER THE NON-STATIONARY RANDOM PROCESS, $f(t)$, WHICH HAS AN UNDERLYING PROBABILITY DENSITY OF $p_I(x;t)$:

$$p_I(x;t) = \begin{cases} \frac{1}{t}\,\mathrm{rect}\left[\frac{x - t/2}{t}\right] ; & |t| \leq 1 \Rightarrow \\ t^2\,\mathrm{rect}\left[\frac{x - 1/2t^2}{1/t^2}\right] ; & |t| > 1 \Rightarrow \end{cases}$$



NOTE, AS REQUIRED, $f(t) \geq 0 \;\forall\, t$.
SUBJECT TO THE UNDERLYING DENSITIES, ONE MAY PLACE ANY CORRELATION OR BAND-WIDTH CONSTRAINTS ON $f(t)$ THAT HE DESIRES.

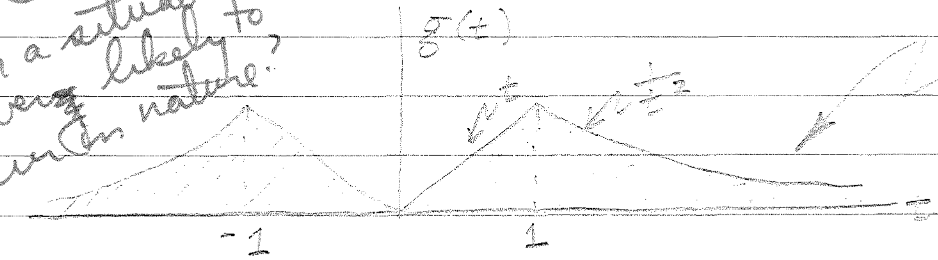
FROM THE STATED DENSITY FUNCTIONS, ONE SEES THAT $f(t)$ IS BOUNDED:

$$0 \leq f(t) \leq \underset{\sim}{g}(t)$$

WHERE, FROM THE DENSITIES,

$$\underset{\sim}{g}(t) = \begin{cases} t ; & |t| \leq 1 \\ 1/t^2 ; & |t| > 1 \end{cases}$$

Bob,
Very interesting but I wonder if such a situation is very likely to occur in nature?



ALL $f(t)$ MUST LIE WITHIN THE SHADED AREA

SINCE $g(t)$ UPPERBOUNDS ALL $f(t)$;

$\int_{-\infty}^{\infty} g(t)\,dt = 7$ UPPERBOUNDS ALL $\int_{-\infty}^{\infty} f(t)\,dt$.

THUS

$$\int_{-\infty}^{\infty} f(t)\,dt \leq 7$$

AND, FINALLY, WE HAVE

$$\langle f(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\,dt = 0 \quad \forall \; f(t)$$

QED*

RETURNING NOW TO THE PROBLEM, A SUFFICIENT CONDITION FOR NON-UNPOLARIZED WOULD BE THAT THE RANDOM PROCESS, $A(t)$, WAS STATIONARY.

* IN GENERAL, IF $p_I(x;t) = \frac{1}{g(t)} \mathrm{rect}\left[\frac{x - g(t)/2}{g(t)}\right]$

THEN $f(t) \geq 0 \;\; \forall \; t$ AND $\int_{-\infty}^{\infty} f(t)\,dt \leq \int_{-\infty}^{\infty} g(t)\,dt \;\; \forall \; f(t)$

$\therefore$ IF $\int_{-\infty}^{\infty} g(t)\,dt < \infty$, THEN $\langle f(t) \rangle = 0 \;\; \forall \; f(t)$

ONE MAY OR MAY NOT RESTRICT $g(t) \geq 0$

4. a. $u(t) = e^{-j[2\pi\nu_0 t - \theta(t)]}$

10

$$\Gamma_u(t_1, t_2) = \langle u^*(t_1)\, u(t_2) \rangle$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{-j 2\pi\nu_0 \tau}\, e^{j[\theta(t+\tau) - \theta(t)]}\, dt$$

WHERE $\tau = t_2 - t_1$, $t = t_1$

$$\Gamma_u(t_1, t_2) = e^{-j 2\pi\nu_0 \tau} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{j[\theta(t+\tau) - \theta(t)]}\, dt$$

$$= e^{-j 2\pi \nu_0 \tau} \left\langle e^{j[\theta(t+\tau) - \theta(t)]} \right\rangle$$

DEFINE $\theta(t) = 2\pi \int_{-\infty}^{t} \nu_p(\xi)\, d\xi$

$\ni$ $\nu_p(\xi)$ IS ERGODIC RANDOM PROCESS

(THUS NOT DEPENDING ON CHOICE OF ORIGIN i.e. ITS ALSO STATIONARY)

THEN

$$\theta(t+\tau) - \theta(t) = 2\pi \int_{t}^{t+\tau} \nu_p(\xi)\, d\xi$$

BUT INTEGRAL LIMITS DEPEND ONLY ON INTERVAL τ (SINCE $\nu_p(t)$ IS STATIONARY). THUS THE PROCESS

$\Delta\theta = \theta(t+\tau) - \theta(t)$ IS STATIONARY

AND THEN, FURTHER ASSUMING ERGODICITY:

$$\left\langle e^{j[\theta(t+\tau) - \theta(t)]} \right\rangle = \left\langle e^{j\Delta\theta} \right\rangle = \overline{e^{j\Delta\theta}}$$

BUT SINCE $\overline{e^{j\Delta\theta\omega}} = M_{\Delta\theta}(\omega)$

WE HAVE $\overline{e^{j\Delta\theta}} = M_{\Delta\theta}(1)$

AND

$$\Gamma_u(t_1, t_2) = e^{-j 2\pi \nu_0 \tau} M_{\Delta\theta}(1)$$

b. FROM PART (A) WE HAVE

$$M_{\Delta\theta}(1) = \overline{e^{j\Delta\theta}} = e^{j2\pi\int_t^{t+\tau} v(t)\,dt}$$

GIVEN THAT $v(t)$ IS ZERO-MEAN GAUSSIAN AND $\theta(t)$ IS GAUSSIAN, ONE MAY ARGUE THAT

$$\theta(t+\tau) - \theta(t) = 2\pi\int_t^{t+\tau} v(t)\,dt$$

IS ZERO MEAN GAUSSIAN. THE CHARACTERISTIC FUNCTION OF A ZERO-MEAN GAUSSIAN PROCESS, $\Delta\theta$, WITH VARIANCE σ^2 IS

$$M_{\Delta\theta}(\omega) = e^{-\frac{1}{2}(\sigma\omega)^2}$$

THUS

$$M_{\Delta\theta}(1) = e^{-\frac{1}{2}\sigma^2}$$

FOR THE ZERO-MEAN GAUSSIAN PROCESS

$$\Delta\theta(\tau) = \theta(t+\tau) - \theta(t)$$

THE CORRESPONDING VARIANCE IS

$$\sigma^2 = E\left[\{\theta(t+\tau) - \theta(t)\}^2\right]$$

THUS

$$M_{\Delta\theta}(1) = e^{-\frac{1}{2}\overline{[\theta(t+\tau) - \theta(t)]^2}}$$

BUT THE EXPONENT IS THE STRUCTURE FUNCTION OF θ:

$$D_\theta(\tau) = E\left[\{\theta(t+\tau) - \theta(t)\}^2\right]$$

THUS

$$M_{\Delta\theta}(1) = e^{-\frac{1}{2}D_\theta(\tau)}$$

AND FINALLY

$$\Gamma(\tau) = e^{-j2\pi\nu_0\tau}\, e^{-\frac{1}{2}D_\theta(\tau)} \checkmark$$

b. CONSIDER THE EXPRESSION

$$M_{\Delta\theta}(1) = \overline{e^{j\Delta\theta}}$$

$$= \overline{e^{j\left[2\pi\int_t^{t+\tau} v_r(\xi)\,d\xi\right]}}$$

$$= \overline{e^{j[\theta(t+\tau)-\theta(t)]}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \overline{\left[j\{\theta(t+\tau)-\theta(t)\}\right]^n}$$

IF $v_r(t)$ IS ZERO MEAN GAUSSIAN, (AND $\theta(t)$ IS GAUSSIAN) IT FOLLOWS THAT $\Delta\theta$ IS ZERO MEAN GAUSSIAN.

THUS, $M_{\Delta\theta}$ IS AN EXPANSION OF THE $\mathcal{F}^{-1}(\;)$ OF A ZERO-MEAN GAUSSIAN DISTRIBUTION.

BUT ALL ODD MOMENTS OF A GAUSSIAN (ZERO MEAN) DISTRIBUTION ARE ZERO. THAT IS

$$\overline{[\theta(t+\tau)-\theta(t)]^{2p+1}} = 0$$

THUS WE MAY REWRITE THE EXPANSION AS

$$M_{\Delta\theta}(1) = \sum_{p=0}^{\infty} \frac{1}{p!}(-1)^p \left\{\overline{[\theta(t+\tau)-\theta(t)]^2}\right\}^p$$

$$= e^{-\frac{1}{2}\overline{[\theta(t+\tau)-\theta(t)]^2}}$$

$$= e^{\frac{1}{2}D_\theta(\tau)}$$

10

5. OMITTING THE OBLIQUITY FACTOR, $\chi(\theta)$, WE START WITH

$$\underline{u}_T(P_0,t) = \iint_\Sigma \frac{2}{jcr}\int_0^\infty \nu\, \underline{U}_T(P_1,\nu)\, e^{-j2\pi\nu(t-r/c)}\, d\nu\, ds$$

SINCE $\Delta\nu \ll \bar{\nu}$, OVER THE INTERVAL OF INTEGRATION, WE HAVE $\nu \simeq \bar{\nu}$. THUS

$$\underline{u}_T(P_0,t) \cong \iint_\Sigma \frac{2}{jcr\bar{\nu}}\int_0^\infty \underline{U}_T(P_1,\nu)\, e^{+j2\pi\nu r/c}\, e^{-j2\pi\nu t}\, d\nu\, ds$$

ONE MUST BE MORE CAREFUL IN THE EXPONENT DUE TO THE COMPARATIVELY LARGE CHANGE IN $e^{j2\pi\nu r/c}$ WITH SMALL CHANGES IN ν. DEFINE $\nu = \bar{\nu} + \delta\nu$.

CONSIDERING THE INTERVAL OF INTEGRATION, WE HAVE $-\frac{\Delta\nu}{2} \le \delta\nu \le \frac{\Delta\nu}{2}$, AND OUR RELATIONSHIP BECOMES

$$u_T(P_0,t) \simeq \iint_\Sigma \frac{1}{jcr\bar{\nu}}\int_0^\infty 2\underline{U}_T(P_1,\nu)\, e^{j2\pi\delta\nu r/c}\, e^{-j2\pi\nu t}\, d\nu \times e^{+j2\pi\bar{\nu} r/c}\, ds$$

SINCE $r \ll \frac{c}{\Delta\nu}$, WE HAVE $\frac{r\Delta\nu}{c} \ll 1$.

THUS $\frac{r\delta\nu}{c} \ll 1$ AND WE MAY SAFELY SET $e^{j2\pi\delta\nu r/c}$ TO UNITY. THIS LEAVES:

$$\underline{u}_T(P_0,t) = \iint_\Sigma \frac{1}{jcr\bar{\nu}}\int_0^\infty 2\underline{U}_T(P_1,\nu)\, e^{-j2\pi\nu t}\, d\nu\, e^{j2\pi\bar{\nu} r/c}\, ds$$

THE INTEGRAL OVER ν IS RECOGNIZED AS AN INVERSE FOURIER TRANSFORM* WHICH BECOMES THE ANALYTIC SIGNAL $\underline{u}_T(P_1,t)$:

$$\underline{u}_T(P_0,t) = \iint_\Sigma \frac{1}{jcr\bar{\nu}}\, \underline{u}_T(P_1,t)\, e^{j2\pi\bar{\nu} r/c}\, ds$$

SETTING $\frac{c}{\bar{\lambda}} = c\bar{\nu}$ GIVES THE DESIRED ANSWER

$$\underline{u}_T(P_0,t) = \iint_\Sigma \frac{e^{j2\pi r/\bar{\lambda}}}{j\bar{\lambda} r}\, \underline{u}(P_1,t)\, ds$$

ONE MAY ALSO GO THROUGH THE SAME STEPS EMPLOYING THE OBIQUITY FACTOR, TO GIVE

$$\underline{u}_T(P_0,t) = \iint_\Sigma \frac{1}{j\bar{\lambda} r}\, e^{j2\pi r/\bar{\lambda}}\, \chi(\theta)\, \underline{u}(P_1,t)\, ds$$

*NOTE COMPATIBILITY WITH EQUATION ON PG. 5.

6.Q. DIVY UP THE ANALYTIC SIGNAL, $U_N(t)$,

10 RESULTING FROM N INDEPENDENTLY OSCILLATING EQUAL-STRENGTH MODES INTO

$$\underline{U}_N(t) = \underline{U}_{N-1}(t) + U_1(t)$$

THEN

$$I_N = I_{N-1} + I_1 + 2\,\mathrm{Re}\, \underline{U}_1(t)\, \underline{U}_{N-1}^*(t)$$

WHERE $I_j = |\underline{U}_j|^2$, $j = N, N-1, 1$

$\underline{U}_j$ MAY BE THOUGHT OF AS A SUM OF PHASERS WITH AMPLITUDE $\sqrt{I_j}$ AND DIRECTION θ_j. IN THAT ALL THE MODES ARE EQUAL STRENGTH, AND CONSIDERING THE CONSTRAINTS PLACED ON THE ANALYTIC SIGNAL (Pg. 36 on);

$$\underline{U}_j = \sqrt{I_j}\, e^{i\theta_j}$$

WHERE θ_j IS UNIFORMLY DISTRIBUTED AND IS INDEPENDENT OF I_j. THUS

$$I_N = I_{N-1} + I_1 + 2\,\mathrm{Re}\sqrt{I_1 I_{N-1}}\, \exp i(\theta_{N-1} - \theta_1)$$

$$= I_{N-1} + I_1 + 2\sqrt{I_1 I_{N-1}}\, \cos\psi$$

WHERE $\psi = \theta_{N-1} - \theta_1$ MOD 2π, IS UNIFORMLY DISTRIBUTED ON $-\pi$ TO π AND IS INDEPENDENT OF I_j. THIS RELATIONSHIP IS SIMPLY A "LAW OF COSINES"

b. SINCE WE HAVE N EQUAL STRENGTH INDEPENDENT MODES

$$\overline{I_N} = N\overline{I_i}$$

WHERE $\overline{I_i}$ IS THE EXPECTED VALUE OF INTENSITY OF A SINGLE MODE. SIMILARLY

$$\overline{I_{N-1}} = (N-1)\,\overline{I_i}$$

THUS, SOLVING FOR $\overline{I_i}$:

$$\overline{I_{N-1}} = \frac{N-1}{N}\,\overline{I_N}$$

C. FROM PART 6a

$$I_N = I_{N-1} + I_1 + 2\sqrt{I_1 I_{N-1}}\cos\psi$$

$$\Rightarrow \overline{I_N^2} = \sigma_N^2 + \overline{I}_N^2$$

$$= E\left[\left\{I_{N-1} + I_1 + 2\sqrt{I_1 I_{N-1}}\cos\psi\right\}^2\right]$$

EXPANDING & RECOGNIZING THAT $\overline{\cos\psi} = 0$

$$\sigma_N^2 + \overline{I}_N^2 = \overline{I_{N-1}^2} + \overline{I_1^2} + 4\overline{I_1 I_{N-1}}\,\overline{\cos^2\psi} + 2\overline{I}_1\overline{I}_{N-1}$$

$$= \sigma_{N-1}^2 + \overline{I}_{N-1}^2 + \sigma_1^2 + \overline{I}_1^2 +$$

$$+ 2\overline{I}_1\overline{I}_{N-1}\left[1 + 2\overline{\cos^2\psi}\right]$$

BUT $\sigma_1^2 = 0$

$$\overline{I}_1 = \frac{1}{N}\overline{I}_N$$

$$\overline{I}_{N-1} = \left(\frac{N-1}{N}\right)\overline{I}_N$$

$$\overline{\cos^2\psi} = \frac{1}{2}$$

$$\Rightarrow \sigma_N^2 + \overline{I}_N^2 = \sigma_{N-1}^2 + \left(\frac{N-1}{N}\right)^2\overline{I}_N^2 + \frac{1}{N^2}\overline{I}_N^2$$

$$+ 4\frac{1}{N}\overline{I}_N\left(\frac{N-1}{N}\right)\overline{I}_N$$

$$\sigma_N^2 = \sigma_{N-1}^2 + \left[\left(\frac{N-1}{N}\right)^2 + \frac{1}{N^2} + \frac{4(N-1)}{N^2} - 1\right]\overline{I}_N^2$$

$$\Rightarrow \frac{\sigma_N}{\overline{I}_N} = \left[\left(\frac{\sigma_{N-1}}{\overline{I}_N}\right)^2 + \left(\frac{N-1}{N}\right)^2 + \frac{1}{N^2} + \frac{4(N-1)}{N^2} - 1\right]^{1/2}$$

[illegible]

The pinholes have diameter d and spacing [illegible]. The [illegible] [illegible] of frequency [illegible], f is the focal length of the lens. [illegible] the fringes [illegible] [illegible]:

(1) The fringe [illegible] of the [illegible]

(2) The fringe [illegible] of the [illegible]

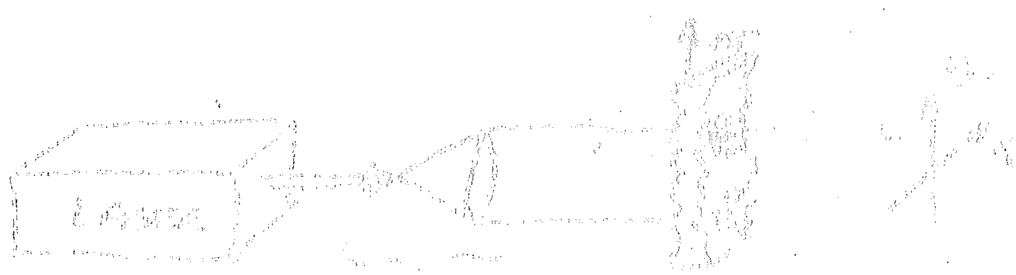
[illegible]

4. Prove that any monochromatic source is perfectly coherent (in the first-order sense).

5. A source is called cross-spectrally pure if its complex degree of coherence factors into a product of temporal and spatial coherence;

$$\Gamma_{12}(\tau) = J_{12}\,\gamma(\tau).$$

Consider the light generated with the output of a CW laser oscillator (assumed initially monochromatic) falls upon a rotating diffuser (ground glass) (as shown illustrated):



The diffuser D is moving in the y-direction with constant velocity v, and therefore its amplitude transmittance may be represented by $\underline{t}(x, y - vt)$. It is known that the statistical autocorrelation function of the random process $\underline{t}(x, y)$ is Gaussian,

$$\underline{\Gamma}_t(\Delta x, \Delta y) = \exp\left\{-a\left[(\Delta x)^2 + (\Delta y)^2\right]\right\}.$$

(a) Is the light transmitted at points P_1 and P_2 cross-spectrally pure?

(b) Repeat with two diffusers, statistically independent and with autocorrelation functions as above, moving in <u>opposite</u> directions, i.e. having a combined amplitude transmittance

BOB MARKS
STAT OPTICS
DUE 2/23/76

10

1. a. $\hat{\mathcal{G}}(\nu) = \frac{1}{N} \sum_{n=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} \delta[\nu - (\bar{\nu} - n\Delta\nu)]$

$$\underline{\gamma}(\tau) = \int_0^\infty \hat{\mathcal{G}}(\nu)\, e^{-j2\pi\nu\tau}\, d\nu$$

$$= \frac{1}{N} \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} \int_0^\infty e^{-j2\pi\nu\tau}\, \delta[\nu - (\bar{\nu} - n\Delta\nu)]\, d\nu$$

SINCE $\bar{\nu} + \frac{(N-1)}{2}\Delta\nu > 0$

$$\underline{\gamma}(\tau) = \frac{1}{N} \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} e^{-j2\pi(\bar{\nu} - n\Delta\nu)\tau}$$

$$= \frac{1}{N} e^{-j2\pi\bar{\nu}\tau} \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} e^{+j2\pi n\Delta\nu\cdot\tau}$$

A GEOMETRIC SERIES WITH TWIXT ?

r = RATIO TWIXT ADJACENT TERMS $= e^{-j2\pi n\tau}$

N = # OF TERMS (ODD)

a_0 = FIRST TERM $= e^{-j2\pi\left[\frac{N-1}{2}\right]\Delta\nu\cdot\tau}$

$$\Rightarrow \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} e^{j2\pi n\Delta\nu\cdot\tau} = a_0 \frac{1 - r^n}{1 - r}$$

$$= e^{-j2\pi\left[\frac{N-1}{2}\right]\Delta\nu\cdot\tau} \frac{1 - e^{+j2\pi N\Delta\nu\tau}}{1 - e^{j2\pi\Delta\nu\cdot\tau}}$$

$$= e^{-j2\pi\left[\frac{N-1}{2}\right]\Delta\nu\cdot\tau} \frac{e^{+j\pi N\Delta\nu\tau}}{e^{j\pi\Delta\nu\tau}} \left[\frac{e^{-j\pi N\Delta\nu\tau} - e^{j\pi N\Delta\nu\tau}}{e^{-j\pi\Delta\nu\tau} - e^{j\pi\Delta\nu\tau}}\right]$$

$$= e^{-j2\pi\left[\frac{N-1}{2}\right]\Delta\nu\cdot\tau} e^{j\pi(N-1)\Delta\nu\tau} \frac{\sin\pi N\Delta\nu\cdot\tau}{\sin\pi\Delta\nu\cdot\tau}$$

$$= \frac{\sin\pi N\Delta\nu\cdot\tau}{\sin\pi\Delta\nu\cdot\tau}$$

✓

$$\Rightarrow \underline{\gamma}(\tau) = \frac{1}{N} e^{-j2\pi\bar{\nu}\tau} \frac{\sin\pi N\Delta\nu\cdot\tau}{\sin\pi\Delta\nu\cdot\tau} \Rightarrow \gamma(\tau) = |\underline{\gamma}(\tau)| = \left|\frac{\sin\pi N\Delta\nu\tau}{N\sin\pi\Delta\nu\tau}\right|$$

b. USE HP-25 WITH PROGRAM:

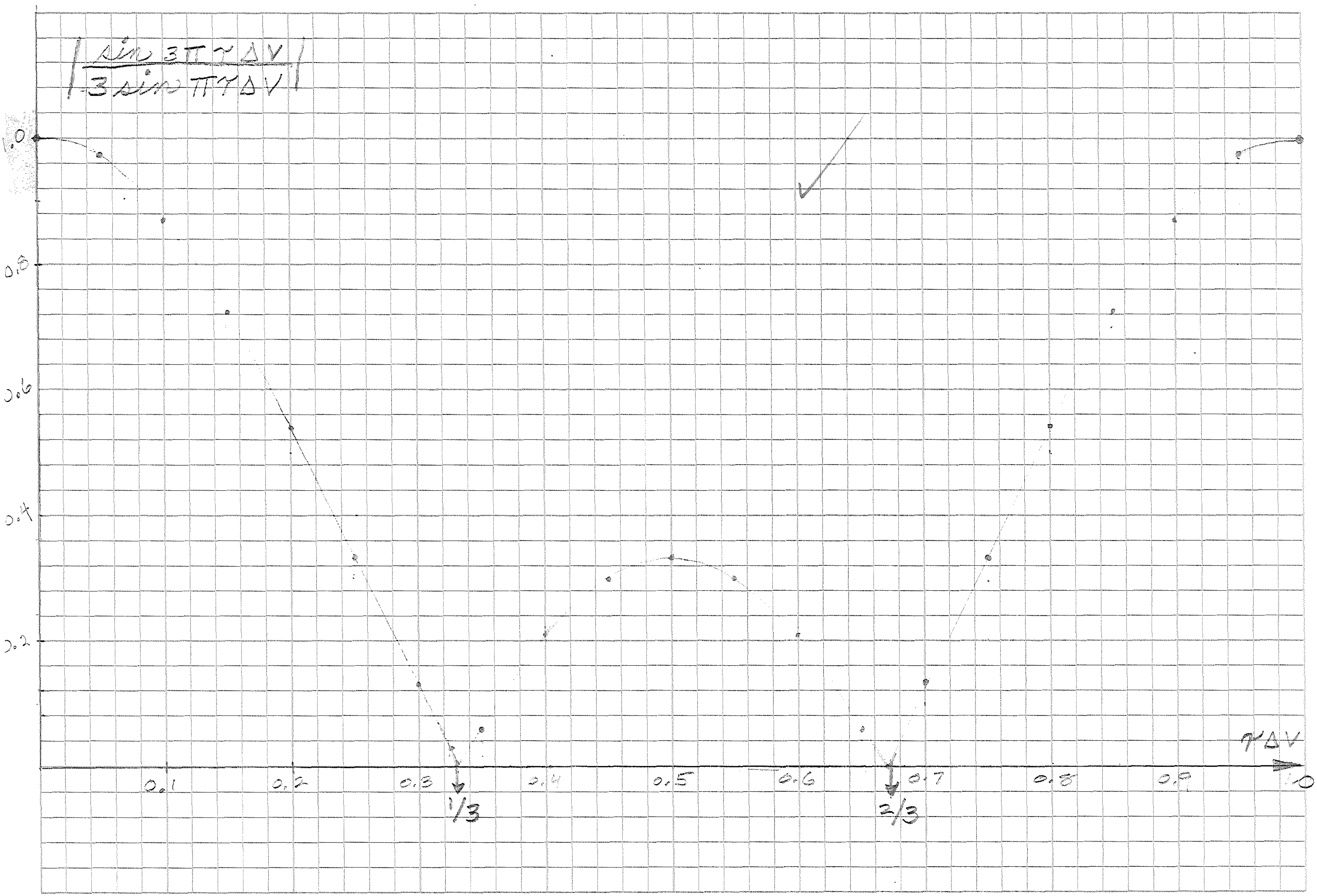
```
(ΔVτ)
π
STO 1
3
×
SIN
RCL 1
SIN
3
×
÷
ABS
GTO 00
```

WILL GIVE $\left|\dfrac{\sin 3\pi(\Delta\nu\tau)}{3\sin\pi(\Delta\nu\tau)}\right| = \gamma(\tau;\Delta\nu)\Big|_{N=3}$

NOTE

$$\gamma(\tau;\Delta\nu)\Big|_{N=3} = 0 \quad \text{FOR} \quad \tau\Delta\nu = \tfrac{1}{3}, \tfrac{2}{3}$$

$$\text{PERIOD (IN } \tau\Delta\nu\text{) OF } \gamma(\tau;\Delta\nu)\Big|_{N=3} = 1$$



$\left|\frac{\sin 3\pi\tau\Delta\nu}{3\sin\pi\tau\Delta\nu}\right|$
1.0
0.8
0.6
0.4
0.2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1/3
2/3
$\tau\Delta\nu$

10

2A. $\Delta V = 1.5 \times 10^9$ HZ

DOPPLER BROADENED $\Rightarrow$ GAUSSIAN STATISTIC

BY GOODMAN'S DEF:

$$\Rightarrow \tau_c = \sqrt{\frac{2 \ln 2}{\pi}} \frac{1}{\Delta V}$$

$$= \sqrt{\frac{2 \ln 2}{\pi}} \frac{10^{-9}}{1.5} = 4.43 \times 10^{-10} \text{ SEC}$$

$$\therefore \ l_c = c\tau_c = (3 \times 10^{10} \tfrac{cm}{SEC}) \tau_c$$

$$= 13.3 \text{ cm}$$

b. SIMILARLY

$$\tau_c = \sqrt{\frac{2 \ln 2}{\pi}} \frac{10^{-9}}{7.5} = 8.86 \times 10^{-11} \text{ SEC}$$

$$l_c = c\tau_c = 2.66 \text{ cm}$$

3.

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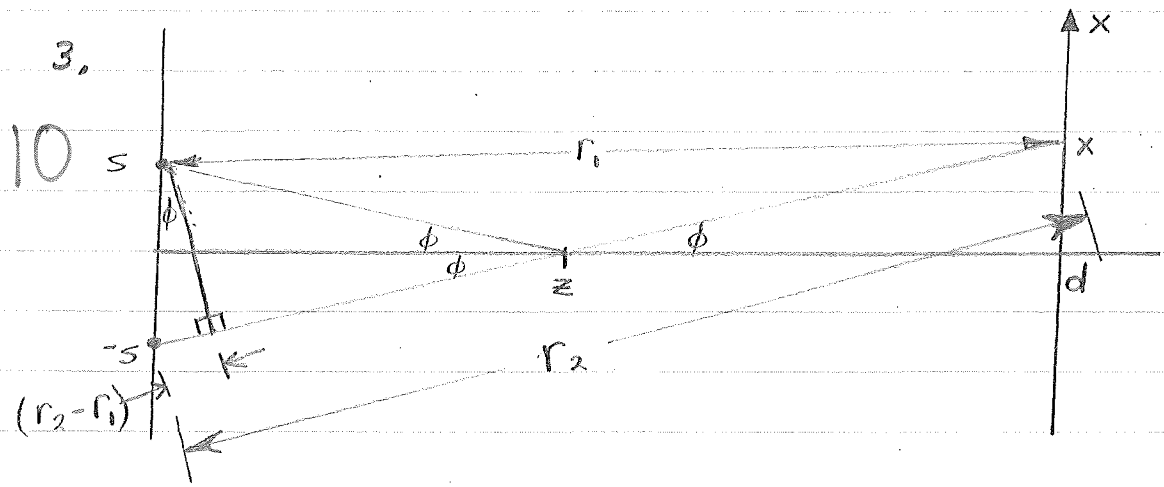


USING PARAXIAL APPROXIMATION (Pg 87)

$$r_2 - r_1 = \frac{2sx}{d} \quad ; \; x>0,\; s>0$$

NOW, (ASSUMING NO LOSSES)

$$I(x) = \langle |U_s(t) + U_{-s}(t + \tfrac{r_2-r_1}{c})|^2 \rangle$$

BUT $U_{-s}(t) = -U_s(t)$ (ie, A SIGN CHANGE)

$$\Rightarrow \quad I = \langle |U_s(t) - U_s(t + \tfrac{r_2-r_1}{c})|^2 \rangle$$

$$= 2I_0 - \langle U_s(t)\,U_s^*(t+\tfrac{r_2-r_1}{c})\rangle - \langle U_s^*(t)\,U_s(t+\tfrac{r_2-r_1}{c})\rangle$$

$$= 2I_0 - 2\,\mathrm{Re}\,\underline{\Gamma}\left(\tfrac{r_2-r_1}{c}\right)$$

WHERE $I_0 = \langle |U_s(t)|^2\rangle = \langle |U_s(t+\tfrac{r_2-r_1}{c})|^2\rangle$

$$\underline{\Gamma}(\tau) = \langle U_s(t)\,U_s^*(t+\tau)\rangle \quad ; \quad \Gamma(0) = I_0$$

$$\Rightarrow \underline{\gamma}(\tau) = \Gamma(\tau)/\underline{\Gamma}(0) = \underline{\Gamma}(\tau)/I_0 \Rightarrow \underline{\Gamma}(\tau) = I_0\,\underline{\gamma}(\tau)$$

THEN, FOR $\tau = \frac{r_2-r_1}{c} \cong \frac{2sx}{dc}$

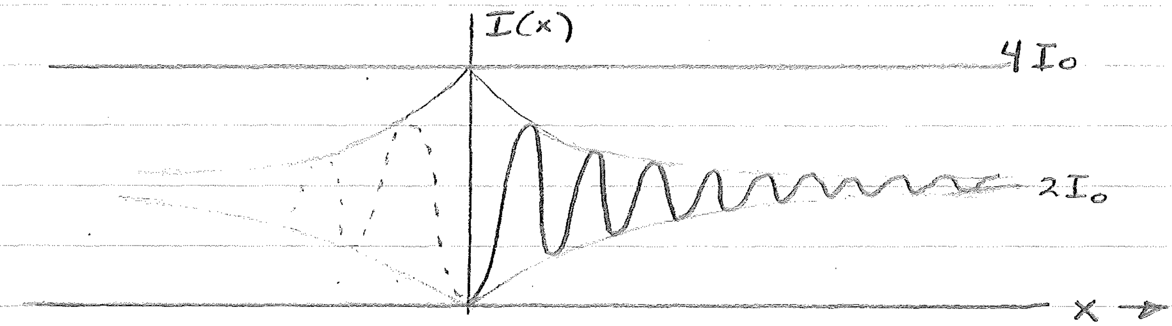
$$I = 2I_0 - 2I_0\,\mathrm{Re}\,\underline{\gamma}(\tau) = 2I_0\left(1 - \mathrm{Re}\,\underline{\gamma}(\tau)\right)$$

WE ARE GIVEN THAT

$$\underline{\gamma}(\tau) = e^{-\pi\Delta\nu|\tau|}\,e^{-j2\pi\bar{\nu}\tau}$$

$$\Rightarrow I = 2I_0\left(1 - e^{-\pi\Delta\nu|\tau|}\cos 2\pi\bar{\nu}\tau\right)$$

$$= 2I_0\left(1 - e^{-\frac{2\pi\Delta\nu s}{dc}|x|}\cos\frac{4\pi\bar{\nu}s}{dc}x\right)$$



a. SPATIAL FREQUENCY, f_o, GIVEN FROM COSINE TERM:

$$f_o = \frac{2\bar{\nu}S}{dc} \qquad \omega_o = 2\pi f_o = \frac{4\pi\bar{\nu}S}{dc}$$

b. FRINGE VISIBILITY:

$$\mathcal{V}(x) = \mathrm{Re}\,\underline{\gamma}(\tau)$$

$$= e^{-\frac{2\pi\Delta\nu S}{dc}|x|}$$

4. FOR BROADBAND LIGHT (SEC. 1.1.2), THE HUYGENS-FRESNEL INTEGRAL MAY BE WRITTEN AS

$$u(P_0,t) = \iint_\Sigma \frac{\frac{d}{dt}\underline{u}(P, t-\frac{r}{c})}{2\pi c r} \chi(\theta) ds$$

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USING PINHOLES @ P_1 AND P_2 (WITH RESPECTIVE ARES A_1 AND A_2), WE HAVE $ds \sim A_i$ AND $\Sigma = P_1 \,\&\, P_2$:

$$u(Q_1, t) = \tilde{K}_1 \frac{d}{dt}\underline{u}(P_1, t-\frac{r_1}{c}) + \tilde{K}_2 \frac{d}{dt} u(P_2, t-\frac{r_2}{c})$$

WHERE, OBVIOUSLY

$$\tilde{K}_1 = \frac{\chi(\theta_1)}{2\pi c r_1} A_1 \quad ; \quad \tilde{K}_2 = \frac{\chi(\theta_2)}{2\pi c r_2} A_2$$

BOTH $\tilde{K}_1$ AND $\tilde{K}_2$ ARE REAL.

NOW

$$\begin{aligned} I(Q) &= \langle |u(Q_1, t)|^2 \rangle \\ &= \langle | \tilde{K}_1 \frac{d}{dt}\underline{u}(P_1, t-\frac{r_1}{c}) + \tilde{K}_2 \frac{d}{dt}\underline{u}(P_2, t-\frac{r_2}{c})|^2 \rangle \\ &= I^{(1)}(Q) + I^{(2)}(Q) + K_1K_2 \langle \frac{d}{dt}\underline{u}(P_1, t-\frac{r_1}{c}) \frac{d}{dt}\underline{U}^*(P_2, t-\frac{r_2}{c}) \rangle \\ &\quad + K_1K_2 \langle \frac{d}{dt}\underline{U}^*(P_1, t-\frac{r_1}{c}) \frac{d}{dt}\underline{u}(P_2, t-\frac{r_2}{c}) \rangle \\ &= I^{(1)}(Q) + I^{(2)}(Q) + K_1K_2 \langle \frac{d}{dt}\underline{U}(P_1, t+\frac{r_2-r_1}{c}) \frac{d}{dt}U^*(P_2, t) \rangle \\ &\quad + K_1K_2 \langle \frac{d}{dt}\underline{U}^*(P_1, t+\frac{r_2-r_1}{c}) \frac{d}{dt}\underline{U}(P_2, t) \rangle \end{aligned}$$

(LAST STEP ASSUMES STATIONARITY)

WHERE

$$I^{(1)}(Q) = K_1^2 \langle |\frac{d}{dt}\underline{u}(P_1, t-\frac{r_1}{c})|^2 \rangle$$

AND

$$I^{(2)}(Q) = K_2^2 \langle |\frac{d}{dt}\underline{u}(P_2, t-\frac{r_2}{c})|^2 \rangle$$

NOW

$$I(Q) = I^{(1)} + I^{(2)} + K_1 K_2 \left\langle \frac{d}{dt} U_1(t+\tau) \frac{d}{dt} U_2^*(t) \right\rangle + K_1 K_2 \left\langle \frac{d}{dt} U_1^*(t+\tau) \frac{d}{dt} U_2(t) \right\rangle$$

WHERE $U_i(\xi) = U(P_i, t)$ AND $\tau = \frac{r_2 - r_1}{c}$

SINCE $\frac{d}{dt} F(t \pm \tau) = \pm \frac{\delta}{\delta\tau} F(t \pm \tau)$

$$I(Q) = I^{(1)} + I^{(2)} + K_1 K_2 \left\langle \frac{\delta}{\delta\tau} U_1(t+\tau) \frac{d}{dt} U_2^*(t) \right\rangle + K_1 K_2 \left\langle +\frac{\delta}{\delta\tau} U_1^*(t+\tau) \frac{d}{dt} U_2(t) \right\rangle$$

$$= I^{(1)} + I^{(2)} + K_1 K_2 \frac{\delta}{\delta\tau} \left\langle U_1(t+\tau) \frac{\delta}{\delta t} U_2^*(t) \right\rangle + K_1 K_2 \frac{\delta}{\delta\tau} \left\langle U_1^*(t+\tau) \frac{d}{dt} U_2(t) \right\rangle$$

SINCE U_i ARE JOINTLY STATIONARY;

$$I(Q) = I^{(1)} + I^{(2)} + K_1 K_2 \frac{\delta}{\delta\tau} \left\langle U_1(t) \frac{\delta}{\delta t} U_2^*(t-\tau) \right\rangle + K_1 K_2 \frac{\delta}{\delta\tau} \left\langle U_1^*(t) \frac{\delta}{\delta t} U_2(t-\tau) \right\rangle$$

$$= I^{(1)} + I^{(2)} + K_1 K_2 \frac{\delta}{\delta\tau} \left\langle U_1(t) \left(-\frac{\delta}{\delta\tau}\right) U_2^*(t-\tau) \right\rangle + K_1 K_2 \frac{\delta}{\delta\tau} \left\langle U_1^*(t) \left(-\frac{\delta}{\delta\tau}\right) U_2(t+\tau) \right\rangle$$

$$= I^{(1)} + I^{(2)} - K_1 K_2 \frac{\delta^2}{\delta\tau^2} \left\langle U_1(t) U_2^*(t+\tau) \right\rangle - K_1 K_2 \frac{\delta^2}{\delta\tau^2} \left\langle U_1^*(t) U_2(t+\tau) \right\rangle$$

NOW $\Gamma_{12}(\tau) = \langle U_1(t) U_2^*(t-\tau) \rangle$ (pg. 22)

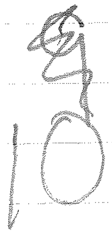
$$\Rightarrow I(Q) = I^{(1)} + I^{(2)} - K_1 K_2 \frac{\delta^2}{\delta\tau^2} \Gamma_{12}(\tau) - K_1 K_2 \frac{\delta^2}{\delta\tau^2} \Gamma^*(\tau)$$

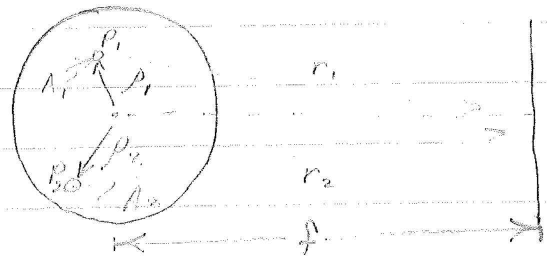
$$= I^{(1)} + I^{(2)} - K_1 K_2 \, Re\left[\frac{\delta^2}{\delta\tau^2} \Gamma(\tau)\right]$$

$$= I^{(1)}(Q) + I^{(2)}(Q) - K_1 K_2 \frac{\delta}{\delta\tau^2} Re\left[\Gamma(\tau)\right] \Big|_{\tau = \frac{r_2 - r_1}{c}}$$

5. FOR QUASI-MONOCHROMATIC LIGHT

$$U(P_0,t) = \int_\Sigma\int \frac{1}{j\lambda r}\, U(P_1, t - \tfrac{r}{c})\, \chi(\theta)\, ds$$





CALL THE LIGHT BEHIND THE LENS $U_B(P_i)$

$$\Rightarrow U(P_i, t) = U_B(P_i, t)\, e^{-j\frac{\pi}{\lambda f}\rho_i^2}$$

MAKING THE USUAL APPROXIMATION:

$$U(Q,t) = K_1\, U_B(P_1, t - \tfrac{r_1}{c})\, e^{-j\frac{\pi}{\lambda f}\rho_1^2}$$
$$+ K_2\, U_B(P_2, t - \tfrac{r_2}{c})\, e^{-j\frac{\pi}{\lambda f}\rho_2^2}$$

$$K_1 = \frac{1}{j\lambda r_1}\chi_1(\theta)A_1 \quad ; \quad K_2 = \frac{1}{j\lambda r_2}\chi_2(\theta)A_2$$

LET $I(Q) = \langle |U(Q,t)|^2 \rangle$

$$= I^{(1)}(Q) + I^{(2)}(Q)$$
$$+ K_1K_2^*\langle U_B(P_1, t - \tfrac{r_1}{c})\, U_B^*(P_2, t - \tfrac{r_2}{c})\rangle e^{-j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$
$$+ K_2K_1^*\langle U_B^*(P_1, t - \tfrac{r_1}{c})\, U_B(P_2, t - \tfrac{r_2}{c})\rangle e^{j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$

Bob – why not approximate in terms of $\frac{r_2 - r_1}{c}$, $\Delta\xi, \Delta\eta$, ρ_2^2, ρ_1^2

WHERE $I^{(i)}(Q) = \langle |U_{B_i}(t)|^2 \rangle$

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q)$$
$$+ K_1K_2^*\langle U_{B_1}(t + \tfrac{r_2 - r_1}{c})\, U_{B_2}^*(t)\rangle e^{-j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$
$$+ K_2K_1^*\langle U_{B_1}^*(t + \tfrac{r_2 - r_1}{c})\, U_{B_2}(t)\rangle e^{+j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$

$$\Gamma_{12}(\tau) = \langle U_1(t+\tau)\, U^*(t)\rangle$$

$$\Rightarrow I(Q) = I^{(1)}(Q) + I^{(2)}(Q)$$
$$+ K_1K_2^*\,\Gamma(\tfrac{r_2 - r_1}{c})\, e^{-j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$
$$+ K_2K_1^*\,\Gamma^*(\tfrac{r_2 - r_1}{c})\, e^{+j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$

(see soln.)

$$= I^{(1)}(Q) + I^{(2)}Q$$
$$+ 2\,\mathrm{Re}\, K_1K_2^*\,\Gamma(\tfrac{r_2 - r_1}{c})\, e^{-j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}$$

(CONT →)

FOR QUASI-MONOCHROMATIC LIGHT

$$\Gamma_{12}(\tau) \cong J_{12}\, e^{-j 2\pi \bar{\nu} \tau}$$

$$\Rightarrow I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2|K_1 K_2^*| J_{12}\, Re\left[e^{-j 2\pi \bar{\nu}\left(\frac{r_2 - r_1}{c}\right)} e^{-j \frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)} \right]$$

$J_{12} = |J_{12}|$; $\alpha_{12} = \arg \underline{J}_{12}$

UNDER THE PARAXIAL APPROXIMATION, THIS BECOMES (pg 96)

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2|K_1 K_2^*| J_{12} \cos\left[\frac{2\pi}{\lambda f}(\Delta\xi X + \Delta\eta Y) - \alpha_{12} + \frac{\pi}{\lambda f}(\rho_2^2 - \rho_1^2) - \frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)\right]$$

$$= I^{(1)}(Q) + I^{(2)}(Q) + 2|K_1 K_2^*| J_{12} \cos\left[\frac{2\pi}{\lambda f}(\Delta\xi X + \Delta\eta Y) - \alpha_{12} + \frac{2\pi}{\lambda f}(\rho_2^2 - \rho_1^2)\right] \quad (1)$$

HMMMM. THE ρ_i'S SHOULD HAVE DROPPED OUT, BUT DIDN'T. IT'S MY CONJECTURE THAT THE LENS TRANSMITANCE SHOULD BE (AS ON pg 81 OF GOODMAN FOURIER OPTICS): $e^{-j\frac{\pi}{\lambda f}\rho^2}$. IF THIS IS TRUE, THEN THE CORRESPONDING ANSWER WOULD BE

See the soln. for the explanation they do drop out

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2|K_1 K_2^*| \cos\left[\frac{2\pi}{\lambda f}(\Delta\xi X + \Delta\eta Y) - \alpha_{12}\right]$$

IN WHICH CASE THE INTENSITY PATTERN WOULD ONLY DEPEND ON THE VECTOR SEPARATION $(\Delta\xi, \Delta\eta)$ OF THE TWO INPUT POINTS. IF THIS CONJECTURE IS INCORRECT, PLEASE POINT OUT MISTAKE.

WE WOULD HAVE GOTTEN Eq. 1 BY USING THE GIVEN TRANSMITTANCE & USING THE Γ_{12}^* FOR Γ_{12}.

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6. SUPPOSE THAT THE LIGHT INTO THE DETECTOR IS $\underline{U}(t)$. THE CORRESPONDING AUTOCORRELATION IS

$$\Gamma(\tau) = \Gamma_{IN}(\tau) = \langle \underline{U}(t+\tau)\,\underline{U}^*(t) \rangle$$

THE LIGHT FALLING ON THE DETECTOR IS (pg. 56)

$$\underline{U}_D(t) = K_1 \underline{U}(t) + K_2 \underline{U}\left(t + \frac{2h}{c}\right)$$

THE CORRESPONDING AUTO-CORRELATION OF THE LIGHT FALLING ON THE DETECTOR IS

$$\begin{aligned}
\Gamma_D(\tau) &= \langle \underline{U}_D(t)\,\underline{U}_D^*(t+\tau) \rangle \\
&= \left\langle \left(K_1 \underline{U}^*(t) + K_2 \underline{U}^*\left(t+\frac{2h}{c}\right)\right)\left(K_1 \underline{U}(t+\tau) + K_2 \underline{U}\left(t+\tau+\frac{2h}{c}\right)\right)\right\rangle \\
&= K_1^2 \langle \underline{U}^*(t)\,\underline{U}(t+\tau) \rangle + K_2^2 \left\langle \underline{U}^*\left(t+\frac{2h}{c}\right)\underline{U}\left(t+\tau+\frac{2h}{c}\right)\right\rangle \\
&\quad + K_1 K_2 \left\langle \underline{U}^*(t)\,\underline{U}\left(t+\tau+\frac{2h}{c}\right)\right\rangle + K_1 K_2 \left\langle \underline{U}^*\left(t+\frac{2h}{c}\right)\underline{U}(t+\tau)\right\rangle \\
&= K_1^2 \langle \underline{U}(t+\tau)\,\underline{U}^*(t) \rangle + K_2^2 \langle \underline{U}(t+\tau)\,\underline{U}^*(t) \rangle \\
&\quad + K_1 K_2 \left\langle \underline{U}\left(t+\tau+\frac{2h}{c}\right)\underline{U}^*(t)\right\rangle + K_1 K_2 \left\langle \underline{U}\left(t+\tau-\frac{2h}{c}\right)\underline{U}^*(t)\right\rangle \\
&= K_1^2 \Gamma(\tau) + K_2^2 \Gamma(\tau) + K_1 K_2 \Gamma\left(\tau+\frac{2h}{c}\right) + K_1 K_2 \Gamma\left(\tau-\frac{2h}{c}\right)
\end{aligned}$$

NOW $\quad \Gamma(\tau) = 4\int_0^\infty \mathcal{G}^{(r,r)}(\nu)\, e^{-j2\pi\nu\tau}\, d\nu$

$$\Rightarrow \mathcal{G}^{(r,r)}(\nu) \propto \mathcal{F}[\Gamma(\tau)]$$

$$\mathcal{G}_D(\nu) = (K_1^2 + K_2^2)\,\mathcal{G}(\nu) + K_1 K_2 \left[\mathcal{G}(\nu)\, e^{-j2\pi\left(\frac{2h}{c}\right)\nu} + \mathcal{G}(\nu)\, e^{j2\pi\left(\frac{2h}{c}\right)\nu}\right]$$

WHERE $\mathcal{G}_D(\nu) = \frac{1}{4}\mathcal{F}[\Gamma_D(\tau)]$; $\mathcal{G}(\nu) = \mathcal{F}\left[\frac{1}{4}\Gamma(\tau)\right]$

$$\mathcal{G}_D(\nu) = \left[(K_1^2 + K_2^2) + 2K_1K_2 \cos 2\pi\left(\frac{2h}{c}\right)\nu\right]\mathcal{G}(\nu) \checkmark$$

a. FOR SMALL VALUES OF $\frac{2h}{c}$ (IN COMPARISON WITH THE RECIPRICAL DIMENSION OF THE "SPREAD" OF $\mathcal{G}(\nu)$] WE HAVE $\mathcal{G}_D(\nu)$ PROPORTIONAL TO $\mathcal{G}(\nu)$ (AND THUS $\hat{\mathcal{G}}(\nu)$ TO $\hat{\mathcal{G}}(\nu)$). WE ALSO MUST ASSUME THAT $\mathcal{G}(\nu)$ LIES WITHIN A POSITIVE "PEAK" OF THE COSINE TERM. AS h INCREASES

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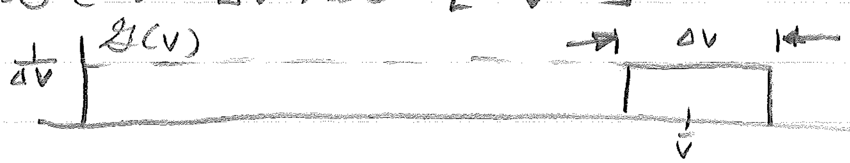
SO DOES THE FREQUENCY OF THE COSINE TERM. THUS, IN THE REGION OF INTEREST, $\mathcal{G}(\nu)$ MIGHT BE SEVERELY DISTORTED. ACTUALLY, FOR SMALL h, $\mathcal{G}(\nu)$ MAY LIE IN THE MINIMUM REGION OF THE COS TERM. THE COS TERM WOULD STILL BE ROUGHLY CONSTANT, BUT ATTENUATION WOULD OCCUR. NOTE THAT $K_1^2+K_2^2+2K_1K_2\cos 2\pi\left(\frac{2h}{c}\right)\nu \geq 0$ (BY SOMEONE'S INEQUALITY). THUS, THE MODULATING TERM CAN'T GO <0. (THIS WOULD DISTORT $\mathcal{G}_D$ BADLY IF $\mathcal{G}$ EXISTED AT THE REGION OF CROSSOVER.

good!

b. $\hat{\mathcal{G}}(\nu) = \frac{1}{\Delta\nu}\, \mathrm{rect}\left[\frac{\nu-\bar{\nu}}{\nu}\right]$



i. $\tau = 0 = \frac{2h}{c}$

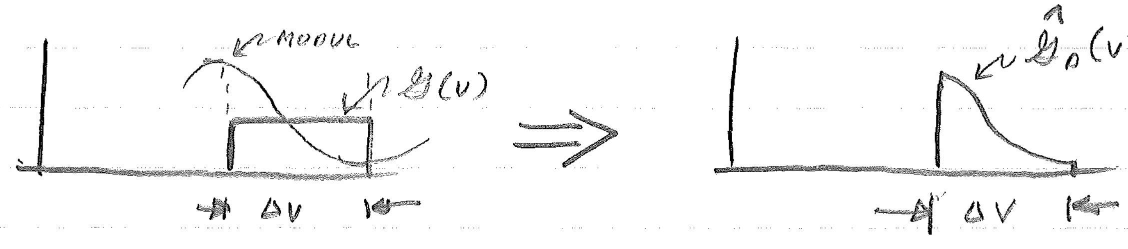
$\Rightarrow \hat{\mathcal{G}}_D(\nu) = C\left[(K_1^2+K_2^2)\right]\hat{\mathcal{G}}(\nu)$ ← STRICTLY PROPORTIONAL OF $\mathcal{G}$'s

(C IS A CONSTANT TO ACCOUNT FOR NORMALIZATION)

ii. $\tau = \frac{1}{2\Delta\nu}$

$\Rightarrow \hat{\mathcal{G}}_D(\nu) = C\left[(K_1^2+K_2^2)+2K_1K_2\cos\frac{\pi\nu}{\Delta\nu}\right]\hat{\mathcal{G}}(\nu)$

MODULATING TERM HAS PERIOD $= 2\Delta\nu$. $\hat{\mathcal{G}}_D(\nu)$ WILL BE NOTICABLY DISTORTED. EXAMPLE:

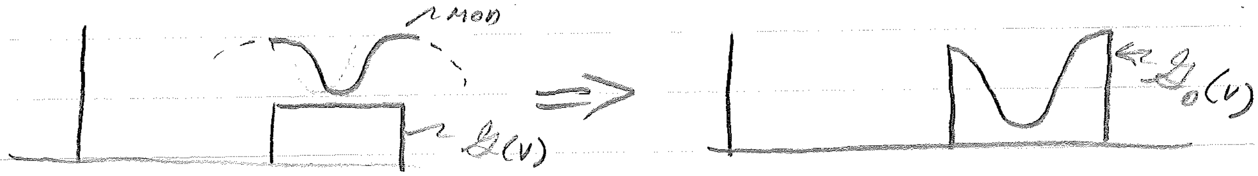


iii. $\tau = \frac{1}{\Delta V}$

$$\Rightarrow \hat{\mathcal{G}}(v) = c\left[(K_1^2 + K_2^2) + 2K_1K_2 \cos \frac{2\pi V}{\Delta V}\right]$$

MODULATING TERM HAS PERIOD OF ΔV.

$\Rightarrow \hat{\mathcal{G}}_D(V)$ WILL BE EVEN MORE DISTORTED: EXAMPLE:



NOTE: AN EXAMPLE h MAY BE GOTTEN BY ASSUMING A $\Delta V \sim 10^9$ Hz AS IN PROBLEM 2.

ii. $\frac{2h}{c} = \frac{1}{\Delta V}$

$$\Rightarrow h = \frac{c}{2\Delta V} = \frac{3 \times 10^{10}}{2 \times 10^9} = 15 \text{ cm}$$

FOR PART iii, $h = 30$ cm

Mon. 2/23

Problem Set #3

1. An idealized model of the (normalized) power spectral density of a gas laser oscillating in N equally spaced modes is

$$\hat{\mathcal{G}}(\nu) = \frac{1}{N} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \delta(\nu - \bar{\nu} - n\Delta\nu)$$

where $\Delta\nu$ is the mode spacing (equal to $c/2\times$ cavity length/velocity of light), $\bar{\nu}$ is the frequency of the central mode, and N has been assumed odd for simplicity.

(a) Show that the corresponding envelope of the complex coherence function is

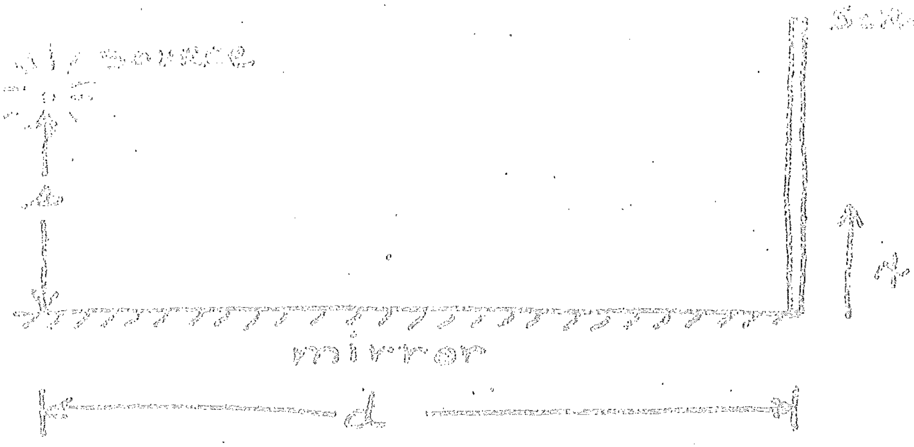
$$\gamma(\tau) = \left| \frac{\sin(N\pi\Delta\nu\tau)}{N\sin(\pi\Delta\nu\tau)} \right|$$

(b) Plot γ vs $\Delta\nu\tau$ for $N=3$ and [illegible]

2. The gas mixture in a HeNe laser (with mirrors removed) emits light at 6328 Å with a doppler broadened spectral width of about 1.5×10^9 Hz. Calculate the coherence time τ_c and the coherence length $l_c = c\tau_c$ (c = velocity of light) for this light. Repeat for the 4880 Å line of the argon ion laser which has a doppler-broadened line width of about 7.5×10^9 Hz. (Both line widths may be assumed to be half-power widths.)

3. (Lloyd's mirror) A point source of light is placed at distance a above a perfectly reflecting mirror. At distance d away interference fringes are observed on a screen (see figure next page). The complex degree of coherence of the light is

$$\gamma(\tau) = e^{-\pi \Delta\nu |\tau|} \, e^{-j2\pi \bar{\nu} \tau}$$



Adopting the assumptions $s \ll d$ and $x \ll d$, and taking account of a sign change of the field upon reflection (polarization assumed parallel with the mirror surface), find:

(a) The spatial frequency of the fringe;

(b) The classical visibility of the fringe as a function of x, assuming equal strength interfering beams.

4. Consider the Young's interference experiment performed with <u>broadband light</u>.

(cont.)

(a) Show that the field incident on the observing screen can be expressed as

$$u(Q,t) = \tilde{K}_1 \frac{d}{dt} u\left(P_1, t-\frac{r_1}{c}\right) + \tilde{K}_2 \frac{d}{dt} u\left(P_2, t-\frac{r_2}{c}\right)$$

where

$$\tilde{K}_i = \iint\limits_{\substack{i\text{th} \\ \text{pinhole}}} \frac{\chi(\theta_i)}{2\pi c r_i}\, da_i \simeq \frac{\chi(\theta_i)\, A_i}{2\pi c r_i}, \quad i=1,2,$$

A_i being the area of the ith pinhole.

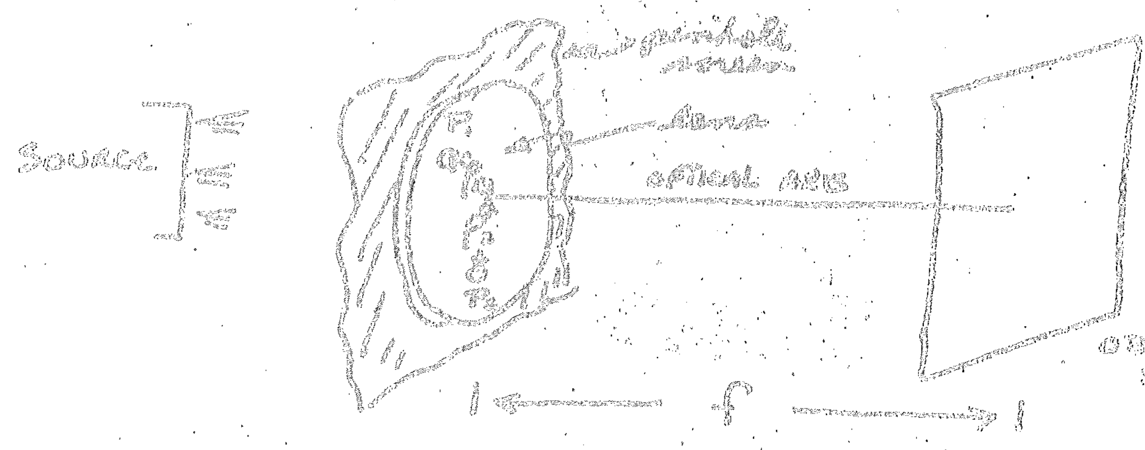
(b) Using the result of part (a), show that the intensity of the light striking the screen can be expressed as

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) - 2\tilde{K}_1\tilde{K}_2 \,\mathrm{Re}\left\{\frac{\partial^2}{\partial \tau^2}\Gamma_{12}\left(\frac{r_2-r_1}{c}\right)\right\}$$

where

$$I^{(i)}(Q) = \tilde{K}_i^2 \left\langle \left| \frac{d}{dt} u\left(P_i, t-\frac{r_i}{c}\right)\right|^2 \right\rangle.$$

5. As shown in the figure below, a positive lens with focal length f is placed in contact with the pinhole screen in Young's interference experiment.



For quasi-monochromatic or narrowband light, the effect of the lens can be modeled by an amplitude transmittance function

$$t_l(\rho) = \exp\left[j\frac{\pi}{\lambda f}\rho^2\right]$$

under paraxial conditions. Show that the spatial phase and period of the fringe pattern depend only on the vector separation of the two pinholes and not on their absolute locations with respect to the optical axis.

6. Consider a Michelson interferometer the as used in Fourier[77] spectroscopy. To obtain high resolution in the computed spectrum, it is necessary that the interferogram be measured out to large path-length differences where the interferogram has fallen to small values.

(a) Show that when large path-length differences exist, the spectrum of the light falling on the detector is significantly different than the spectrum of light entering the interferometer.

(b) If the spectrum of light entering the interferometer is

$$\hat{G}(\nu) = \frac{1}{\Delta\nu}\,\mathrm{rect}\,\frac{\nu-\bar{\nu}}{\Delta\nu},$$

calculate the spectrum of the light falling on the detector when $\tau = 0$, $\frac{1}{2\Delta\nu}$ and $\frac{1}{\Delta\nu}$, τ being the time delay between the two interferometer paths.

EE 5358 (JFW) HW set #2 - Solutions ①

① (a) $\hat{G}(\nu) = \frac{1}{N} \sum_{-\frac{N-1}{2}}^{\frac{N-1}{2}} \delta(\nu - \bar{\nu} + n\Delta\nu)$

∴ from pg. 63 of notes

$$\underline{\gamma}(\tau) = \int_0^\infty \hat{G}(\nu) e^{-j2\pi\nu\tau} d\nu$$

$$= \frac{1}{N} \sum_{-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j2\pi(\bar{\nu} - n\Delta\nu)\tau}$$

where we've used the "sifting" prop. of the Dirac delta fun —

i.e. $\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)$ if $f(x)$ is cont. at $x = a$

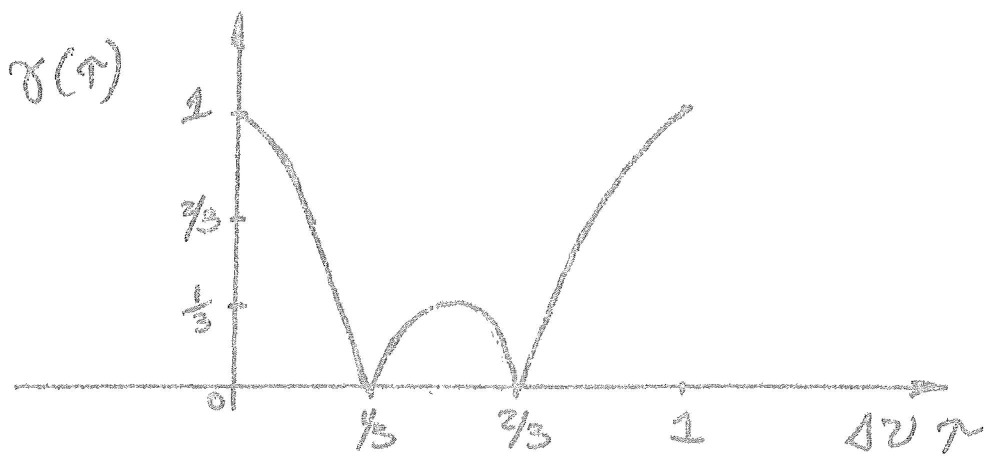
Now $\underline{\gamma}(\tau) = \frac{e^{-j2\pi\bar{\nu}\tau}}{N} e^{-j\frac{2\pi(N-1)}{2}\Delta\nu\tau} \left\{ \frac{1 - e^{j2\pi N\Delta\nu\tau}}{1 - e^{j2\pi\Delta\nu\tau}} \right\}$

$$= \frac{e^{-j2\pi\bar{\nu}\tau}}{N} e^{-j\frac{2\pi(N-1)}{2}\Delta\nu\tau} e^{+j\frac{2\pi(N-1)}{2}\Delta\nu\tau} \left\{ \frac{\frac{e^{+j\pi N\Delta\nu\tau} - e^{-j\pi N\Delta\nu\tau}}{2}}{e^{+j\pi\Delta\nu\tau} - e^{-j\pi\Delta\nu\tau}} \right\}$$

$$\underline{\gamma}(\tau) = \frac{e^{-j2\pi\bar{\nu}\tau}}{N}\left\{\frac{\sin N\pi\Delta\nu\tau}{\sin \pi\Delta\nu\tau}\right\}$$

$$\therefore\ \gamma(\tau) = |\underline{\gamma}(\tau)| = \left|\frac{\sin N\pi\Delta\nu\tau}{N\sin \pi\Delta\nu\tau}\right|$$

(b) For $N=3$ and $0 \le \Delta\nu\tau \le 1$



(2) From pg. 71 of notes.

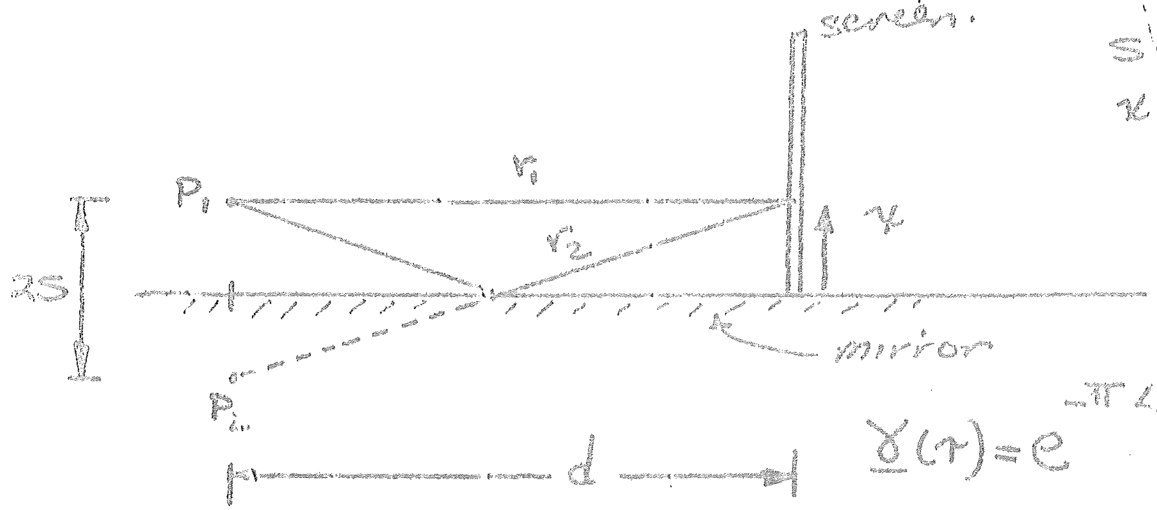
$$\left.\begin{aligned} \tau_c &= \frac{0.664}{\Delta\nu} \\ l_c &= c\,\tau_c \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \tau_c &= \frac{0.664}{1.5\times10^{9}} = \underline{0.443\times10^{-9}\ \text{sec.}} \\ l_c &= (3\times10^{8})(0.443\times10^{-9}) \\ \therefore\ l_c &= \underline{0.133\ \text{meter.}} \end{aligned}\right.$$

HeNe laser.

While for Argon laser

$$\tau_c = \frac{0.664}{7.5\times10^{9}} = \underline{0.088\times10^{-9}\ \text{sec.}},\quad \underline{l_c = 0.0266\ \text{meters.}}$$

③ Lloyd's mirror :



$s << d$
$x << d$

$$\underline{\gamma}(\tau) = e^{-\pi \Delta\nu |\tau|} e^{-j\pi\bar{\nu}\tau}$$

$$\Rightarrow \gamma(\tau) = |\underline{\gamma}(\tau)| = e^{-\pi\Delta\nu|\tau|}$$

This is simply Young's experiment with quasimonochromatic light, however the sign change upon reflection changes the sign of the cosinusoidal interference term. The intensity at the screen is similar to the results on fig 8.5 of the notes.

$$I(x) = I^{(1)} + I^{(2)} - 2\sqrt{I^{(1)}I^{(2)}}\,\gamma(\tau)\,\mathrm{Re}\left\{e^{-j2\pi\bar{\nu}\tau}\right\}$$

$$\cong I^{(1)} + I^{(2)} - 2\sqrt{I^{(1)}I^{(2)}}\,\gamma(\tau)\cos\left[2\pi\bar{\nu}\cdot\frac{2sx}{cd}\right]$$

where we made use of

$$\tau = \frac{r_2 - r_1}{c} = \frac{1}{c}\left\{\sqrt{d^2+(s+x)^2} - \sqrt{d^2+(s-x)^2}\right\} \cong \frac{d}{c}\left\{\left[1 + \frac{(s+x)^2}{2d^2} + \cdots\right] - \left[1 + \frac{(s-x)^2}{2d^2} + \cdots\right]\right\}$$

$$\cong 2sx/cd = \tau$$

ⓐ The spatial freq. of the fringe is $\frac{2s\bar{\nu}}{cd}$ ④

ⓑ $I^{(1)} = I^{(2)}$

$$I(x) = 2I^{(1)}\left[1 - \gamma(\tau)\cos 2\pi\bar{\nu}\cdot\frac{2sx}{dc}\right]$$

$$\mathcal{V} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{[1+\gamma(\tau)] - [1-\gamma(\tau)]}{[1+\gamma(\tau)] + [1-\gamma(\tau)]}$$

$$= \gamma(\tau) = e^{-\pi\Delta\nu\left|\frac{2sx}{cd}\right|}$$

which indicates that the fringe visibility falls off in a negative exponential way with increasing $|x|$ (actually only $x > 0$ is of interest here anyway).

---

④ (a).

$$\underline{U}(Q,t) = \iint_{\Sigma_1} \frac{\frac{d}{dt}\underline{U}(P_1, t - \frac{r_1}{c})}{2\pi c r_1}\chi(\theta_1)\,dS_1 + \iint_{\Sigma_2} \frac{\frac{d}{dt}\underline{U}(P_2, t - \frac{r_2}{c})}{2\pi c r_2}\chi(\theta_2)\,dS_2$$

Σ_1, Σ_2 small

$$\cong \frac{d}{dt}\underline{U}(P_1, t - \frac{r_1}{c})\iint_{\Sigma_1}\frac{\chi(\theta_1)}{2\pi r_1 c}\,dS_1 + \frac{d}{dt}\underline{U}(P_2, t - \frac{r_2}{c})\cdot\iint_{\Sigma_2}\frac{\chi(\theta_2)}{2\pi r_2 c}\,dS_2$$

$$\underline{U}(Q,t) = \tilde{K}_1 \frac{d}{dt}\underline{U}\left(P_1, t-\frac{r_1}{c}\right) + \tilde{K}_2 \frac{d}{dt}\underline{U}\left(P_2, t-\frac{r_2}{c}\right).$$

(b) $I(Q) = \langle \underline{U}^*(Q,t)\,\underline{U}(Q,t)\rangle$

$$\Rightarrow \quad I(Q) = \tilde{K}_1^2 \left\langle \left|\frac{d}{dt}\underline{U}\left(P_1, t-\frac{r_1}{c}\right)\right|^2\right\rangle + \tilde{K}_2^2 \left\langle \left|\frac{d}{dt}\underline{U}\left(P_2, t-\frac{r_2}{c}\right)\right|^2\right\rangle$$

$$+ 2\tilde{K}_1\tilde{K}_2 \,\mathrm{Re}\left\{\left\langle \frac{d}{dt}\underline{U}\left(P_1, t-\frac{r_1}{c}\right)\frac{d}{dt}\underline{U}^*\left(P_2, t-\frac{r_2}{c}\right)\right\rangle\right\}$$

Since we have an ergodic random process,

$$\left\langle \frac{\partial}{\partial t_1}\underline{U}(P_1,t_1)\frac{\partial}{\partial t_2}\underline{U}^*(P_2,t_2)\right\rangle$$

$$= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\langle \underline{U}(P_1,t_1)\underline{U}^*(P_2,t_2)\rangle$$

$$= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\Gamma_{12}(t_1-t_2) = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\Gamma_{12}(\tau)$$

where $\tau \triangleq t_1 - t_2$

$$= \frac{\partial}{\partial(t_1-t_2)}\,\frac{\partial}{-\partial(t_1-t_2)}\Gamma_{12}(\tau) = -\frac{\partial^2}{\partial\tau^2}\Gamma_{12}(\tau)$$

$\boxed{\text{Note:}}$ The argument above holds only in the case of an ergodic process. -i.e.

$\langle x'y'\rangle = R_{x'y'} = E\langle x'(t)\,y'(t+\tau)\rangle$

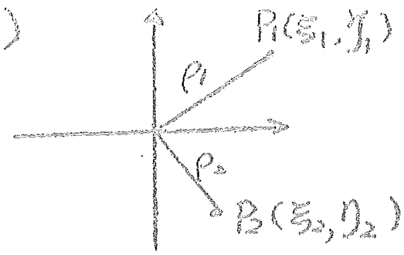
now substitute the just-obtained result into the expression for $I(Q)$ to obtain

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) - 2\tilde{K}_1 \tilde{K}_2 \, \mathrm{Re}\left\{ \frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau) \Big|_{\tau = \frac{r_2 - r_1}{c}} \right\} \quad (6)$$

(5)

Marks

(5)



" ⁎ "

$$r_i \simeq f + \frac{(\xi_i - X)^2}{2f} + \frac{(\eta_i - Y)^2}{2f} = f + \frac{X^2+Y^2}{2f} + \frac{\rho_i^2}{2f} - \frac{\xi_i X + \eta_i Y}{f}$$

$i = 1, 2.$

$$\therefore \frac{r_2 - r_1}{c} \simeq \frac{1}{2fc}\left\{\rho_2^2 - \rho_1^2 + 2\Delta\xi X + 2\Delta\eta Y\right\} \quad ----(1)$$

$\Delta\xi = \xi_1 - \xi_2$

$\Delta\eta = \eta_1 - \eta_2$

Furthermore, we note that the effect of lens is shifting the phase by $\frac{\pi\rho_i^2}{\lambda f}$, that is equivalent to the (forth) shifting on time by $\frac{\pi\rho_i^2}{\lambda f c}$, hence

without lens: $\underline{u}(Q,t) = \underline{K_1}\,\underline{u}(P_1, t - \frac{r_1}{c}) + \underline{K_2}\,\underline{u}(P_2, t - \frac{r_2}{c})$

with lens: $\underline{u}(Q,t) = \underline{K_1}\,\underline{u}(P_1, t - \frac{r_1}{c} + \frac{\rho_1^2}{2cf}) + \underline{K_2}\,\underline{u}(P_2, t - \frac{r_2}{c} + \frac{\rho_2^2}{2cf})$

Using this modification, and carring out the same derivation of sec. 2.2.2. we will obtain [see P. 85]

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2\sqrt{I^{(1)}(Q)\,I^{(2)}(Q)}\;\gamma_{12}\left(\frac{r_2 - r_1}{c} + \frac{\rho_1^2 - \rho_2^2}{2cf}\right)\cos\left[2\pi\bar{\nu}\left(\frac{r_2 - r_1}{c} + \frac{\rho_1^2 - \rho_2^2}{2cf}\right) - \alpha_{12}\left(\frac{r_2 - r_1}{c} + \frac{\rho_1^2 - \rho_2^2}{2cf}\right)\right]$$

From eq. (1) $\frac{r_2 - r_1}{c} + \frac{\rho_1^2 - \rho_2^2}{2cf} \simeq \frac{1}{cf}\left[\Delta\xi X + \Delta\eta Y\right]$

$$I(\theta)=I^{(1)}(\theta)+I^{(2)}(\theta)+2\sqrt{I^{(1)}(\theta)I^{(2)}(\theta)}\;\gamma_{12}\left(\frac{\Delta\xi x+\Delta\eta y}{cf}\right)\cdot$$

$$\cos\left[2\pi\bar{\nu}\left(\frac{\Delta\xi x+\Delta\eta y}{cf}\right)-\alpha_{12}\left(\frac{\Delta\xi x+\Delta\eta y}{cf}\right)\right]$$

Thus the spatial phase (in this case, "$-\alpha_{12}(\cdot)$") and period of the fringe pattern depend only on separation $(\Delta\xi, \Delta\eta)$.

Remark: The spatial period (vector) $\vec{L}$, should satisfy $\vec{L}\cdot\left(\frac{\Delta\xi}{\bar{\lambda}f}, \frac{\Delta\eta}{\bar{\lambda}f}\right)=1$.

(6) $u^D(t) = K_1\, u(t) + K_2\, u\left(t+\frac{2h}{c}\right)$

$\Gamma_D(\tau) = \langle u^D(t)\, u^{D*}(t+\tau)\rangle$ D: detector

$= \langle [K_1 u(t) + K_2 u(t+\frac{2h}{c})][K_1 u^*(t+\tau) + K_2 u^*(t+\tau+\frac{2h}{c})]\rangle$

$= K_1^2\,\Gamma(\tau) + K_2^2\,\Gamma(\tau) + K_1K_2\,\Gamma(\tau+\frac{2h}{c}) + K_1K_2\,\Gamma(\tau-\frac{2h}{c})$

Then we can relate $\mathcal{G}^{(r,r)}(\nu)$, power spectrum on detector, to the original power spectrum

$\mathcal{G}(\nu) = \int_{-\infty}^{\infty} \Gamma(\tau)\, e^{j2\pi\nu\tau}\, d\tau$

i.e. $\mathcal{G}_D^{(r,r)}(\nu) = \int_{-\infty}^{\infty} \Gamma_D(\tau)\, e^{j2\pi\nu\tau}\, d\tau$

[illegible] $\mathcal{G}^{(r,r)}(\nu) + K_1K_2 e^{\ldots}\, \mathcal{G}^{(r,r)}(\nu)$

$$= \left[K_1^2 + K_2^2 + 2K_1K_2\cos\left(4\pi\nu\frac{h}{c}\right)\right]\mathcal{G}^{(r,r)}(\nu)$$

If. $\frac{c}{2h} \sim \Delta\nu$, i.e, if h is not much smaller than $\frac{c}{2\Delta\nu}$

we will get a modulated spectrum [i.e, distortion in frequency domain]

(b) $\mathcal{G}(\nu) = \frac{1}{\Delta\nu}\,\mathrm{rec}\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right) = \mathcal{G}^{(r,r)}(\nu)$ for $\nu > 0$

$$\tau = \frac{2h}{c}, \quad \hat{\mathcal{G}}_D(\nu) = \frac{\mathcal{G}_d(\nu)}{\int_0^\infty \mathcal{G}_d(\nu)\,d\nu} = \frac{[K_1^2+K_2^2+2K_1K_2\cos(2\pi\nu\tau)]\cdot\frac{1}{\Delta\nu}\,\mathrm{rec}\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)}{\int_0^\infty \mathcal{G}_d(\nu)\,d\nu}$$

$$\int_0^\infty \mathcal{G}_d(\nu)\,d\nu = K_1^2+K_2^2+2K_1K_2\int_{\bar{\nu}-\frac{\Delta\nu}{2}}^{\bar{\nu}+\frac{\Delta\nu}{2}}\frac{1}{\Delta\nu}\cos 2\pi\nu\tau\,d\nu$$

$$= K_1^2+K_2^2+\frac{2K_1K_2}{\pi\tau\Delta\nu}\cos(2\pi\tau\bar{\nu})\sin(\pi\tau\Delta\nu)$$

$$\therefore \hat{\mathcal{G}}_d(\nu) = \frac{[K_1^2+K_2^2+2K_1K_2\cos 2\pi\nu\tau]\,\frac{1}{\Delta\nu}\,\mathrm{rec}\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)}{K_1^2+K_2^2+\frac{2K_1K_2}{\pi\tau\Delta\nu}\cos(2\pi\bar{\nu}\tau)\sin(\pi\Delta\nu\tau)}$$

① $\tau = 0, \quad \hat{\mathcal{G}}_d(\nu) = \frac{1}{\Delta\nu}\,\mathrm{rect}\,\frac{\nu-\bar{\nu}}{\Delta\nu}$

② $\tau = \frac{1}{2\Delta\nu} \quad \hat{\mathcal{G}}_d(\nu) = \frac{K_1^2+K_2^2+2K_1K_2\cos(\pi\nu/\Delta\nu)}{K_1^2+K_2^2+\frac{4K_1K_2}{\pi}\cos(\pi\bar{\nu}/\Delta\nu)}\left[\mathrm{rect}\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)\right]\frac{1}{\Delta\nu}$

③ $\tau = \frac{1}{\Delta\nu} \quad \hat{\mathcal{G}}_d(\nu) = \frac{K_1^2+K_2^2+2K_1K_2\cos(2\pi\nu/\Delta\nu)}{K_1^2+K_2^2}\,\frac{1}{\Delta\nu}\,\mathrm{rect}\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)$

$\frac{4}{5}$

① $\underline{u}(Q,t) = \iint_{\Sigma_1} \frac{\frac{d}{dt}\underline{u}(P,t-\frac{r}{c})}{2\pi c r}\chi(\theta)dS$

$\underline{\Gamma}(Q_1, Q_2; \tau) = \langle \underline{u}(Q_1, t+\tau)\underline{u}^*(Q_2, t)\rangle$

$= \left\langle \iint_{\Sigma_1} \frac{\frac{d}{dt}\underline{u}(P_1, t+\tau-\frac{r_1}{c})}{2\pi c r_1}\chi(\theta_1)dS_1 \iint_{\Sigma_1} \frac{\frac{d}{dt}\underline{u}^*(P_2, t-\frac{r_2}{c})}{2\pi c r_2}\chi(\theta_2)dS_2 \right\rangle$

$= \iiiint_{\Sigma_1 \Sigma_1} \frac{\langle \frac{d}{dt}\underline{u}(P_1, t+\tau-\frac{r_1}{c})\frac{d}{dt}\underline{u}^*(P_2, t-\frac{r_2}{c})\rangle}{4\pi^2 c^2 r_1 r_2}\chi(\theta_1)\chi(\theta_2)dS_1 dS_2$

From Prob 4(b) of (# 2), we know that

$\langle \frac{d}{dt}\underline{u}(P_1, t+\tau-\frac{r_1}{c})\frac{d}{dt}\underline{u}^*(P_2, t-\frac{r_2}{c})\rangle = -\frac{\partial^2}{\partial\tau^2}\langle \underline{u}(P_1, t+\tau-\frac{r_1}{c})\underline{u}^*(P_2, t-\frac{r_2}{c})\rangle$

$= -\frac{\partial^2}{\partial\tau^2}\underline{\Gamma}(P_1, P_2; \tau+\frac{r_2-r_1}{c})$

hence

$\underline{\Gamma}(Q_1, Q_2; \tau) = -\iint_{\Sigma_1}\iint_{\Sigma_1} \frac{\partial^2}{\partial\tau^2}\underline{\Gamma}(P_1, P_2; \tau+\frac{r_2-r_1}{c})\frac{\chi(\theta_1)}{2\pi c r_1}\frac{\chi(\theta_2)}{2\pi c r_2}dS_1 dS_2$

② Wave equation for propagation of $\underline{\Gamma}_{12}(\tau)$

$$\nabla_{1,2}^2 \underline{\Gamma}_{12}(\tau) = \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\underline{\Gamma}_{12}(\tau)$$

Quasi-monochromatic light $\underline{\Gamma}_{12}(\tau) = \underline{J}_{12}e^{j2\pi\bar{\nu}\tau}$

substituting, and approx. $\nabla_{1,2}^2 \underline{J}_{12}e^{j2\pi\bar{\nu}\tau}$ by $e^{j2\pi\bar{\nu}\tau}\nabla_{1,2}^2\underline{J}_{12}$

$\frac{\partial^2}{\partial\tau^2}\underline{J}_{12}e^{j2\pi\bar{\nu}\tau}$ by $-(2\pi\bar{\nu})^2\underline{J}_{12}e^{j2\pi\bar{\nu}\tau}$

we get $(\nabla_{1,2}^2 \underline{J}_{12})\, e^{j2\pi\bar{\nu}\tau} = -\left(\frac{2\pi\bar{\nu}}{c}\right)^2 \underline{J}_{12}\, e^{j2\pi\bar{\nu}\tau}$

$$\Rightarrow \quad \nabla_{1,2}^2 \underline{J}_{12} + \bar{k}^2 \underline{J}_{12} = 0, \quad \bar{k} \equiv \frac{2\pi\bar{\nu}}{c} = \frac{2\pi}{\bar{\lambda}}$$

※

③ By refering "Fig 2-7b", we know that, (2) the effect of finite bandwidth "$\Delta\nu$" will cause a "finite" fringe extent approximately equal to $\left(\frac{fc}{\Delta\nu S}\right)$

Furthermore, we know that the transform of a finite-size pinhole, will be an Airy Pattern, and its 1st zero will be $\left(\frac{1.22\bar{\lambda}f}{d}\right)$

Now effect (1) dominates over (2) simply means that

$$\frac{fc}{\Delta\nu S} >> \frac{1.22\bar{\lambda}f}{d}$$

$$\Rightarrow \text{i.e.} \quad \frac{\Delta\nu}{\frac{c}{\bar{\lambda}}} << \frac{d}{1.22S} \Rightarrow \frac{\Delta\nu}{\bar{\nu}} << \frac{d}{1.22S}$$

④ Monochromatic wave $\underline{u}(P,t) = \underline{u}(P)\, e^{-j2\pi\nu_0 t}$ ※

$$\Gamma_{12}(\tau) = \langle \underline{u}(P_1,t)\, \underline{u}^*(P_2,t+\tau) \rangle$$

$$= \underline{u}(P_1)\, \underline{u}^*(P_2) \langle e^{-j2\pi\nu_0 t}\, e^{j2\pi\nu_0(t+\tau)} \rangle$$

$$= \underline{u}(P_1)\, \underline{u}^*(P_2) \langle e^{j2\pi\nu_0\tau} \rangle_{\text{average over } t}$$

$$= \underline{u}(P_1)\, \underline{u}^*(P_2)\, e^{j2\pi\nu_0\tau}$$

$$\therefore |\gamma_{12}(\tau)| = \left| \frac{\Gamma_{12}(\tau)}{[\Gamma_{11}(0)\Gamma_{22}(0)]^{1/2}} \right| = \frac{|u(P_1)|\,|u^*(P_2)|\,|e^{-j2\pi\bar{\nu}\tau}|}{|u(P_1)|\,|u^*(P_2)|} = 1 \quad ※$$

$\therefore$ It is perfectly coherent.

(5) (a) For idealy monochromatic light, $u(P_1,t) = u(P_1)e^{-j2\pi\nu t}$

The output of the diffuser is $\tilde{u}(P_1,t) = u(P_1,t)\,t(x, y-vt)$

$$\Gamma_{12}(\tau) = \langle \tilde{u}(P_1,t+\tau)\tilde{u}^*(P_2,t) \rangle$$

$$= u(P_1)u^*(P_2)\langle e^{-j2\pi\nu(t+\tau)} e^{j2\pi\nu t}\, t(x_1, y_1 - v(t+\tau))\, t^*(x_2, y_2 - vt) \rangle$$

$$= u(P_1)u^*(P_2)\, e^{-j2\pi\nu\tau} \langle t(x_1, y_1 - v(t+\tau))\, t^*(x_2, y_2 - vt) \rangle$$

$$= u(P_1)u^*(P_2)\, e^{-j2\pi\nu\tau} \exp\{-a[(\Delta x)^2 + (\Delta y + v\tau)^2]\}$$

$$= A(P_1,P_2) \exp\{-a[(\Delta x)^2 + (\Delta y)^2]\}\, e^{-j2\pi\nu\tau}\, e^{-a(v\tau)^2}\, \underline{\underline{e^{-2a\Delta y v\tau}}}$$

Note $\gamma_{12}(\tau) = |u(P_1)u^*(P_2)|^{-1}\Gamma_{12}(\tau)$ not separable

$\Rightarrow \therefore$ it is Not cross-spectrally pure.

(b) In this case, output of the diffuser is

$$\tilde{u}(P,t) = u(P,t)\,t_1(x, y-vt)\,t_2(x, y+vt)$$

Carry out the similar procedure,

we will get

$$\Gamma_{12}(\tau) = \underline{u}(P_1)\underline{u}^*(P_2)\, e^{-j2\pi\nu\tau} \langle \underline{t}_1(x_1, y_1 - v(t+\tau))\, \underline{t}_2(x_1, y + v(t+\tau))\,\cdot$$

$$\underline{t}_1^*(x_2, y_2 - vt)\, \underline{t}_2^*(x_2, y_2 + vt) \rangle$$

$$= \underline{u}(P_1)\underline{u}^*(P_2)\, e^{-j2\pi\nu\tau} \langle \underline{t}_1(x_1, y_1 - v(t+\tau))\, \underline{t}_1^*(x_2, y_2 - vt) \rangle \cdot$$

$$\langle \underline{t}_2(x_1, y_1 + v(t+\tau))\, \underline{t}_2^*(x_2, y_2 + vt) \rangle$$

$$= \underline{u}(P_1)\underline{u}^*(P_2)\, e^{-j2\pi\nu\tau}\, \underline{\gamma}_t(\Delta x, \Delta y - v\tau)\, \underline{\gamma}_t(\Delta x, \Delta y + v\tau)$$

$$= \underline{u}(P_1)\underline{u}^*(P_2)\, e^{-j2\pi\nu\tau}\, e^{-a[(\Delta x)^2 + (\Delta y + v\tau)^2]}\, e^{-a[(\Delta x)^2 + (\Delta y - v\tau)^2]}$$

$$= \underline{u}(P_1)\underline{u}^*(P_2)\, e^{-2a[(\Delta x)^2 + (\Delta y)^2]}\, e^{-j2\pi\nu\tau}\, e^{-2a(v\tau)^2}$$

$$= \mu_{12}\, \underline{\gamma}(\tau)$$

$$\Longrightarrow \gamma_{12}(\tau) = |u(P_1)u^*(P_2)|^{-1}\, \Gamma_{12}(\tau) = \mu_{12}\, \underline{\gamma}(\tau)$$

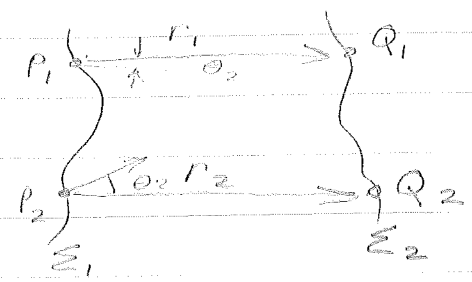
∴ It is cross-spectrally pure.

※.

BOB MARKS
DUE 3-1

10

1.



$$\underline{U}(Q_i) = \iint_{\Sigma_i} \frac{\frac{d}{dt}\underline{U}(P_i, t-\frac{r_i}{c})\,\chi(\theta_i)}{2\pi r c}\, ds$$

$$\Gamma(Q_1, Q_2, \tau) = \langle \underline{U}(Q_1, t+\tau)\, \underline{U}^*(Q_2, t)\rangle$$

$$\underline{U}(Q_1, t) = \iint_{\Sigma_1} \frac{\chi(\theta_1)}{2\pi r_1 c} \frac{d}{dt}\underline{U}(P_1, t-\frac{r_1}{c})\, ds_1$$

$$\underline{U}(Q_2, t) = \iint_{\Sigma_1} \frac{\chi(\theta_2)}{2\pi r_2 c} \frac{d}{dt}\underline{U}(P_2, t-\frac{r_2}{c})\, ds_2$$

$$\underline{U}(Q_1, t+\tau)\,\underline{U}^*(Q_2, t) = \iint_{\Sigma_1}\iint_{\Sigma_1} \frac{\frac{d}{dt}\underline{U}(P_1, t+\tau-\frac{r_1}{c})\,\frac{d}{dt}\underline{U}^*(P_2, t-\frac{r_2}{c})}{2\pi r_2 c \quad 2\pi r_1 c}$$

$$\times \chi(\theta_1)\,\chi(\theta_2)\, ds_1\, ds_2$$

$$\Rightarrow \Gamma(Q_1, Q_2; \tau)$$

$$= \iint_{\Sigma_1}\iint_{\Sigma_1} \frac{\chi(\theta_1)\chi(\theta_2)\,\langle \frac{d}{dt}\underline{U}(P_1, t+\tau-\frac{r_1}{c})\,\frac{d}{dt}\underline{U}^*(P_2, t-\frac{r_2}{c})\rangle}{2\pi r_2 c \quad 2\pi r_1 c} \qquad (I)$$

SINCE $\frac{\delta}{\delta t} F(t \pm \tau) = \pm\frac{\delta}{\delta \tau} F(t\pm\tau)$ AND
SINCE ERGODICITY IS ASSUMED:

$$\langle \tfrac{d}{dt}\underline{U}(P_1, t+\tau-\tfrac{r_1}{c})\,\tfrac{d}{dt}\underline{U}^*(P_2, t-\tfrac{r_2}{c})\rangle = \langle \tfrac{d}{dt}\underline{U}(P_1, t+\tau+\tfrac{r_2-r_1}{c})\,\tfrac{d}{dt}\underline{U}^*(P_2, t)\rangle$$

$$= \langle \tfrac{\delta}{\delta\tau}\underline{U}(P_1, t+\tau+\tfrac{r_2-r_1}{c})\,\tfrac{d}{dt}\underline{U}^*(P_2, t)\rangle$$

$$= \tfrac{\delta}{\delta\tau}\langle \underline{U}(P_1, t+\tau+\tfrac{r_2-r_1}{c})\,\tfrac{d}{dt}\underline{U}^*(P_2, t)\rangle$$

$$= \tfrac{\delta}{\delta\tau}\langle \underline{U}(P_1, t+\tfrac{r_2-r_1}{c})\,\tfrac{d}{dt}\underline{U}^*(P_2, t-\tau)\rangle$$

$$= \tfrac{\delta}{\delta\tau}\langle \underline{U}(P_1, t+\tfrac{r_2-r_1}{c})\,(-\tfrac{\delta}{\delta\tau})\underline{U}^*(P_2, t-\tau)\rangle$$

$$= -\tfrac{\delta^2}{\delta\tau^2}\langle \underline{U}(P_1, t+\tfrac{r_2-r_1}{c})\,\underline{U}^*(P_2, t-\tau)\rangle$$

$$= -\tfrac{\delta^2}{\delta\tau^2}\langle \underline{U}(P_1, t+\tau+\tfrac{r_2-r_1}{c})\,\underline{U}^*(P_2, t)\rangle$$

$$= -\tfrac{\delta^2}{\delta\tau^2}\Gamma(P_1, P_2, \tau+\tfrac{r_2-r_1}{c})$$

WHERE $\Gamma(P_1, P_2; \tau) \triangleq \langle \underline{U}(P_1, t+\tau)\,\underline{U}^*(P_2, t)\rangle$

SUBSTITUTING INTO EQ. 1 GIVES THE FINAL DESIRED RESULT:

$$\underline{\Gamma}(Q_1, Q_2; \tau) = -\iint_{\Sigma_1}\iint_{\Sigma_2} \frac{\delta^2}{\delta\tau^2}\underline{\Gamma}\left(P_1, P_2; \tau + \frac{r_2 - r_1}{c}\right) \frac{\chi(\theta_1)}{2\pi r_1 c}\,\frac{\chi(\theta_2)}{2\pi r_2 c}\, ds_1\, ds_2$$

2. UNDER QUASIMONOCHROMATIC CONDITIONS:

$$\Gamma_{12}(\tau) = \underline{J}_{12}\, e^{-j2\pi\bar{\nu}\tau} \quad \ni \quad \underline{J}_{12} \triangleq \Gamma_{12}(0)$$

10

CONSIDER, FIRST,

$$\nabla_1^2\, \Gamma_{12}(\tau) = \frac{1}{c^2}\frac{\delta^2}{\delta\tau^2}\Gamma_{12}(\tau)$$

THUS

$$e^{-j2\pi\bar{\nu}\tau}\, \nabla_1^2 J_{12} = \frac{1}{c^2} J_{12} \frac{\delta^2}{\delta\tau^2} e^{-j2\pi\bar{\nu}\tau}$$

$$= \frac{1}{c^2} J_{12}\, {}^{-}(2\pi\bar{\nu})^2 e^{-j2\pi\bar{\nu}\tau}$$

$$= {}^{-}\bar{k}^2 J_{12}\, e^{-j2\pi\bar{\nu}\tau} \quad \ni \quad \bar{k} = \frac{2\pi}{\lambda} = \frac{2\pi\bar{\nu}}{c}$$

$$\Rightarrow \nabla_1^2 \underline{J}_{12} + \bar{k}^2 \underline{J}_{12} = 0$$

SIMILARLY: $\nabla_1^2 \underline{J}_{12} + \bar{k}^2 J_{12} = 0$

NOTE: THE HELMOLTZ RELATIONSHIP IS ESSENTIALLY THE TEMPORAL FOURIER TRANSFORM OF THE ORIGINAL DIFFERENTIAL EQUATION RELATIONSHIP. DEFINE

good!

$$\mathcal{F}[g(\tau)] = \int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi\bar{\nu}\tau} d\tau$$

THEN, SINCE

$$\nabla_i^2\, \Gamma_{12}^{(\tau)} = \frac{1}{c^2}\frac{\delta^2}{\delta\tau^2}\Gamma_{12}(\tau)$$

WE HAVE

$$\mathcal{F}[\nabla_i^2 \Gamma_{12}(\tau)] = \frac{1}{c^2}\mathcal{F}\left[\frac{\delta^2}{\delta\tau^2}\Gamma(\tau)\right] \quad ; \quad i = 1, 2$$

$$\nabla_i^2\, \mathcal{F}\{\Gamma_{12}(\tau)\} = \frac{1}{c^2}(-i\,2\pi\bar{\nu})^2\, \mathcal{F}[\Gamma(\tau)]$$

LETTING $J_{12} = \mathcal{F}[\Gamma_{12}(\tau)]$ GIVES

$$\nabla_i^2 J_{12} = {}^{-}\left(\frac{2\pi\bar{\nu}}{c^2}\right) J_{12}$$

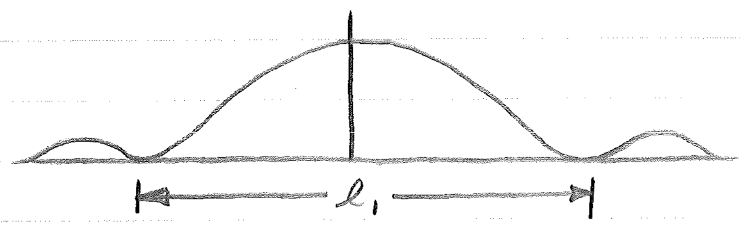
$$= {}^{-}\bar{k}^2 J_{12}$$

OR: $\nabla_i^2 J_{12} + \bar{k}^2 J_{12} = 0$

10

3. TO SOLVE THIS PLOBLEM, ONE MAY LOOK AT FRINGE DAMPING (ie THE FRINGE ENVELOPE) DUE TO ① FINITE PINHOLE SIZE, AND ② FINITE BANDWIDTH.

FOR THE FIRST CASE, THE ENVELOPE IS GIVEN IN Fig 2-9 (WE'LL WON'T HAVE THE PICTURED SHIFTING, THOUGH, DUE TO THE LENS). THE SPACE BETWEEN "NULLS" IS $\ell_1 = \frac{2.44\bar{\lambda} f}{d} = \frac{2.44\, c f}{\bar{\nu} d}$:



FOR CASE ②, THE SPACING BETWEEN "NULLS" IS GIVEN IN FIG. 2.7

$$\ell_2 = {}^{2fc}/_{S\Delta\nu}$$

ACTUALLY, ℓ_2 IS TWICE THE "HALF-WIDTH" OF THE FRINGE ENVELOPE.

IN ORDER FOR THE EFFECT OF FINITE PINHOLES TO DOMINATE OVER THE EFFECT OF FINITE BANDWIDTH, WE MUST REQUIRE THAT $\ell_2 >> \ell_1$, OR

$$\frac{2fc}{S\Delta\nu} >> \frac{2.44\, cf}{\bar{\nu} d}$$

OR

$$\frac{\bar{\nu}}{\Delta\nu} >> \frac{1.22\, S}{d}$$

FOR THE EFFECT OF THE BANDWIDTH TO DOMINATE:

$$\frac{\bar{\nu}}{\Delta\nu} << \frac{1.22\, S}{d}$$

4. THE ANALYTIC SIGNAL ASSOCIATED WITH A MONOCHROMATIC SOURCE IS GIVEN ON pg. 3 AS

10

$$\underset{\sim}{u}(P,t) = \underline{U}(P,\nu)\, e^{-j2\pi\nu t}$$

THEN:

$$\Gamma_{12}(\tau) = \langle u(P_1, t+\tau)\, u^*(P_2, t)\rangle$$

$$= \langle \underline{U}(P_1,\nu)\, e^{-j2\pi\nu(t+\tau)}\, \underline{U}^*(P_2,\nu)\, e^{j2\pi\nu t}\rangle$$

$$= \underline{U}(P_1,\nu)\, \underline{U}^*(P_2,\nu)\, e^{-j2\pi\nu\tau}\langle 1\rangle$$

$$= \underline{U}(P_1,\nu)\, \underline{U}^*(P_2,\nu)\, e^{-j2\pi\nu\tau}$$

$$\underset{\sim}{\gamma}_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{[\Gamma_{11}(0)\,\Gamma_{22}(0)]^{1/2}}$$

$$\Gamma_{ii}(0) = \langle |u(P_i,t)|^2\rangle = |\underline{U}(P_i,\nu)|^2$$

$$\Rightarrow \underset{\sim}{\gamma}_{12} = \frac{\underline{U}(P_1,\nu)\, \underline{U}^*(P_2,\nu)}{|\underline{U}(P_1,\nu)|\,|\underline{U}(P_2,\nu)|}\, e^{-j2\pi\nu\tau}$$

$$|\underset{\sim}{\gamma}_{12}| = \frac{|\underline{U}(P_1,\nu)|\,|\underline{U}(P_2,\nu)|}{|\underline{U}(P_1,\nu)|\,|\underline{U}(P_2,\nu)|}\, |e^{-j2\pi\nu\tau}|$$

$$= 1$$

THIS IS THE STRICTEST DEFINITION OF FULL COHERENCE ON Pg. 117, AND THUS FULLFILLS THE MORE LENIENT DEFINITION ON Pg. 118. ie, ANY MONOCHROMATIC SOURCE IS PERFECTLY COHERENT (BY EITHER DEFINITION).

5. THE MONOCHROMATIC LIGHT INCIDENT ON THE DIFFUSER IS

$$\underline{u}(P_i, t) = \underline{U}(P_i, \nu) e^{-j2\pi \nu t} \quad ; \; i = 1, 2$$

10 THE DIFFUSER TRANSMITTANCE IS

$$\underline{t}(P_i, t) = t(x_i, Y_i - \mathcal{V}t)$$

THE LIGHT FROM THE DIFFUSER IS THUS

$$U_d(P_i, t) = \underline{u}(P_i, t)\underline{t}(P_i, t)$$

NOW

$$\Gamma_{12}(\tau) = \langle u_d(P_1, t+\tau)\, u_d^*(P_2, t) \rangle$$
$$= \underline{U}(P_1, \nu)\underline{U}^*(P_2, \nu) e^{-j2\pi\nu\tau}$$
$$\times \langle t(x_1, Y_1 - \mathcal{V}(t+\tau))\, t^*(x_2, Y_2 - \mathcal{V}t) \rangle$$

NOW

$$\Gamma_{ii}(0) = |\underline{U}_i(P_i, \nu)| \langle |t(x_i, Y_i - \mathcal{V}t)|^2 \rangle$$

THUS

$$\underline{\gamma}_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{[\Gamma_{11}(0)\Gamma_{22}(0)]^{1/2}}$$

$$= \frac{\underline{U}(P_1, \nu)\underline{U}^*(P_2, \nu)}{|\underline{U}(P_1, \nu)||\underline{U}(P_2, \nu)|} \; \frac{\langle t(x_1, Y_1 - \mathcal{V}(t+\tau))t^*(x_2, Y_2 - \mathcal{V}t) \rangle}{[\langle |t(x_1, Y_1 - \mathcal{V}t)|^2 \rangle \langle |t(x_2, Y_2 - \mathcal{V}t|^2 \rangle]^{1/2}}$$

NOW LETS TRY TO GUESS AT WHAT'S MEANT BY

$$\underline{\gamma}_t(\Delta x, \Delta Y) = e^{-a(\Delta x^2 + \Delta Y^2)}$$

LOOK'S LIKE IT SAYS THAT THE SECOND ORDER STATISTICS OF THE DIFFUSER DEPEND ONLY ON DIFFERECE TWIXT SPOTS.

THAT IS

$$\gamma_t(\Delta x, \Delta y) = \frac{E[\underline{t}(x_1+\Delta x, y_1+\Delta x)\,\underline{t}^*(x_1, y_1)]}{[E\{|\underline{t}(x_1, y_1)|^2\} E\{|\underline{t}(x_1+\Delta x, y_1+\Delta y\}]^{1/2}}$$

ASSUMING ERGODICITY (IN THE SPATIAL SENSE), WE WRITE

$$\underline{\gamma}_{12} = \frac{\underline{U}_1, \underline{U}_2^*}{|\underline{U}_1\ \underline{U}_2|}\ \gamma_T(\Delta x, \Delta y - v\tau) \quad ; \Delta x = x_1 - x_2 \quad \Delta y = y_1 - y_2$$

?

DON'T LOOK CROSS-SPECTRALLY PURE:

$$\underline{\gamma}_{12} = \frac{\underline{U}_1 \underline{U}_2^*}{|\underline{U}_1 \underline{U}_2|}\ e^{-a(\Delta x^2 + (v\tau - \Delta y)^2)}$$

b) FOR TWO DIFFUSERS, WE HAVE (BY INSPECTION)

$$\underline{\gamma}_{12}(\tau) = \frac{\underline{U}_1 \underline{U}_2^*}{|\underline{U}_1\ \underline{U}_2|}\ \frac{\langle t_1(x_1, y_1 - v(t+\tau))\ t_1(x_2, y_2 - vt)\rangle}{[\langle |t_1(x_1, y_1 - vt)|^2\rangle \langle |t_1(x_2, y_2 - vt)|^2\rangle]^{\frac{1}{2}}}$$

$$\times \frac{\langle t_2(x_1, y_1 + v(t+\tau))\ t_2(x_2, y_2 + vt)\rangle}{[\langle |t_2(x_1, y_1 + vt)|^2\rangle \langle |t_2(x_2, y_2 + vt|^2\rangle]^{\frac{1}{2}}}$$

WE MAY SEPARATE THEM LIKE THIS DUE TO STATISTICAL INDEPENDENCE. NOW

$$\underline{\gamma}_{12}(\tau) = \frac{\underline{U}_1 \underline{U}_2^*}{|\underline{U}_1 \underline{U}_2|}\ \underline{\gamma}_t(\Delta x, \Delta y - v\tau)\ \underline{\gamma}_t(\Delta x, \Delta y + v\tau)$$

$$= \frac{\underline{U}_1 \underline{U}_2^*}{|\underline{U}_1 \underline{U}_2|}\ e^{-2a\Delta x^2}\ e^{-a(\Delta y - v\tau)^2}\ e^{-a(\Delta y + v\tau)^2}$$

$$= \frac{\Psi_1 \Psi_2^*}{|\Psi_1 \Psi_2|} e^{-2a\Delta X^2} e^{-2a\Delta Y^2} e^{-2a(\nu\tau)^2}$$

SO HERE, THE LIGHT IS CROSS-SPECTRALLY PURE WITH

$$\mu_{12} = \frac{\Psi_1 \Psi_2^*}{|\Psi_1 \Psi_2|} e^{-2a(\Delta X^2 + \Delta Y^2)}$$

AND $\gamma(\tau) = e^{-2a(\nu\tau)^2}$

Marks.

EE5328 Due 3/8/76

PROBLEM SET #~~5~~ 4

#1. The sun subtends an angle of about 32 minutes of arc (.0093 radians) on earth. Assuming a mean wavelength of 5500 Å, calculate the separation of two pinholes for which Young's fringes first vanish, the experiment being performed on earth with filtered (quasi-monochromatic) sunlight.

#2. A one millimeter pinhole is placed immediately in front of an incoherent source. The light passed by the pinhole is to be used in a diffraction experiment for which it is desired to illuminate a distant 1mm aperture coherently. Given $\lambda = 5000$ Å, calculate the minimum distance between the pinhole source and the diffracting aperture.

#3. Consider an incoherent source radiating with spatial intensity distribution $I(\xi,\eta)$.

(a) Using the Van Cittert - Zernike theorem and Parseval's theorem of Fourier analysis, show that the coherence area of the light (mean wavelength $\bar{\lambda}$) at distance z from the source can be expressed as

$$A_c = (\bar{\lambda} z)^2 \frac{\iint_{-\infty}^{\infty} I^2(\xi,\eta)\, d\xi\, d\eta}{\left[\iint_{-\infty}^{\infty} I(\xi,\eta)\, d\xi\, d\eta\right]^2}$$

(b) Show that for any incoherent source with intensity distribution describable as

$$I(\xi,\eta) = I_0 P(\xi,\eta)$$

where P takes on values of 0 or 1 only, then

$$A_c = \frac{(\lambda z)^2}{A_s}$$

where A_s is the area of the source.

#4. Show that if an incoherent quasi-monochromatic source is placed in the front focal plane of a positive lens, the complex coherence factor in the rear focal plane is given by

$$\mu(\Delta x, \Delta y) = \frac{\iint_{-\infty}^{\infty} I(\xi,\eta)\, e^{+j\frac{2\pi}{\lambda F}(\Delta x\xi + \Delta y\eta)}\, d\xi\, d\eta}{\iint_{-\infty}^{\infty} I(\xi,\eta)\, d\xi\, d\eta}$$

Pg: 150
SUB. 1,26

where $I(\xi,\eta)$ represents the intensity distribution in the front focal plane, F is the focal length, and the phase factor ψ of the usual Van Cittert-Zernike theorem is identically zero.

#5. Find the complex coherence factor μ_{12} for

(a) a monochromatic spherical wave diverging from a point on the optical axis at distance z from the plane containing P_1 and P_2. (Use a paraxial approximation)

(b) a monochromatic laser beam with a plane wavefront but with intensity cross section

$$I(x,y) = I_0 \exp\left\{-\frac{x^2+y^2}{W^2}\right\}$$

under the assumption of normal incidence on the plane containing P_1 and P_2.

(c) an incoherent quasi-monochromatic ring source of annular shape with outer radius a_2 and inner radius a_1.

PROBLEM SET #5 (CONT.)

#6. It is desired to use a Michelson stellar interferometer to determine the brightness of two components of a twin star. The individual components are known to be uniformly bright circular disks. Their angular diameters α and β, and their angular separation γ, are all known. We also know that $\gamma \gg \alpha$, $\gamma \gg \beta$. How could we determine their relative brightness I_α / I_β from measurements of $|\mu_{12}(a)|$ with the interferometer?

Problem Set 5 due

Thursday, May 8

KS

From the table [illegible] 3, we [illegible] that the visibility of fringes [illegible] corresponding to [illegible] is the zero visibility corresponding to [illegible] zero of the Bessel function, we have

$$\beta = 1.22 \frac{\bar{\lambda}}{\theta} = \frac{1.22 \times 5500 \times 10^{-10}}{0.0093}, [illegible]$$

The answer to this problem is actually subject to how "coherent" we require it to be. One possible answer is merely considering the so-called "coherence area" A_c, [illegible]

$$A_c = \frac{(\bar{\lambda} z)^2}{A_s} > \pi \left(\frac{10^{-3}}{2}\right)^2, \quad A_c = A_s = \pi r^2 \text{ in the [illegible]}$$

Hence we want $z^2 > \frac{(\pi r^2)^2}{\bar{\lambda}^2}$

$$\Rightarrow z > \frac{\pi r^2}{\bar{\lambda}} = \frac{\pi \left(\frac{1}{2} 10^{-3}\right)^2}{5500 \times 10^{-10}} = [illegible] \text{ m} \quad \#$$

The Van Cittert-Zernike theorem.

$$\mu(\Delta X, \Delta Y) = \frac{e^{-j\psi} \iint I(\xi,\eta)\, e^{j\frac{2\pi}{\bar{\lambda} z}(\Delta X \xi + \Delta Y \eta)}\, d\xi\, d\eta}{\iint I(\xi,\eta)\, d\xi\, d\eta}$$

$$= \frac{e^{-j\psi}}{\iint I(\xi,\eta)\, d\xi\, d\eta} \mathcal{F}\{I\} \left(\frac{\Delta X}{\bar{\lambda} z}, \frac{\Delta Y}{\bar{\lambda} z}\right)$$

[illegible]

$$= (\bar{\lambda} z)^2 \left[\iint I(\xi,\eta)\, d\xi\, d\eta\right]^{-2} \iint_{-\infty}^{\infty} [illegible]$$

$$= (\bar{\lambda} z)^2 \left[\iint I(\xi,\eta)\, d\xi\, d\eta\right]^{-2} \iint_{-\infty}^{\infty} I^2(\xi,\eta)\, d\xi\, d\eta$$

and
$$A_c = \frac{(\bar{\lambda} z)^2 \iint I^2(\xi,\eta)\, d\xi\, d\eta}{\left[\iint I(\xi,\eta)\, d\xi\, d\eta\right]^2}$$

(c) For $I(\xi,\eta) = I_0 P(\xi,\eta)$

$$\iint_{-\infty}^{\infty} I(\xi,\eta)\, d\xi\, d\eta = \iint_{A_s} I_0\, d\xi\, d\eta = A_s I_0$$

$$\iint_{-\infty}^{\infty} I^2(\xi,\eta)\, d\xi\, d\eta = \iint_{A_s} I_0^2\, d\xi\, d\eta = A_s I_0^2$$

$$\therefore A_c = (\bar{\lambda} z)^2 \frac{A_s I_0^2}{(A_s I_0)^2} = \frac{(\bar{\lambda} z)^2}{A_s}$$

(d) From the boxed equation of P. 180

$$J_f(x_1, y_1; x_2, y_2) = \frac{1}{(\bar{\lambda} f)^2} \iiiint_{-\infty}^{\infty} J_o(\xi_1, \eta_1; \xi_2, \eta_2) \cdot$$

$$\exp\left[-j \frac{2\pi}{\bar{\lambda} f}(x_1 \xi_1 + y_1 \eta_1 - x_2 \xi_2 - y_2 \eta_2)\right] d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2$$

For incoherent source, we can simplify the above equation by the fact $J_o(\xi_1, \eta_1; \xi_2, \eta_2) = \kappa I(\xi_1, \eta_1) \delta(\xi_1 - \xi_2, \eta_1 - \eta_2)$

Thus we obtain

[illegible]

Then $\mu_{12} = \dfrac{J_{12}}{[J_{11} J_{22}]^{1/2}}$

Hence we get

$$\mu(\Delta X, \Delta Y) = \frac{\iint I(\xi,\eta)\, e^{j\frac{2\pi}{\lambda z}(\Delta X \xi + \Delta Y \eta)}\, d\xi\, d\eta}{\iint I(\xi,\eta)\, d\xi\, d\eta}$$

(b) $A(P_1) = \dfrac{e^{j\frac{2\pi}{\lambda} r_1}}{r_1}$, $\quad u(P_1) = A(P_1)\, e^{j\phi(t)}$

$$J_{12}(P_1,P_2) = \langle u(P_1)\, u^*(P_2) \rangle = \left\langle \frac{e^{j\frac{2\pi}{\lambda} r_1}\, e^{-j\frac{2\pi}{\lambda} r_2}}{r_1 r_2} \right\rangle$$

$$= \frac{e^{j\frac{2\pi}{\lambda}(r_1 - r_2)}}{z^2} \simeq \frac{e^{j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2)}}{z^2}$$

$$\left(\because\ r_i \simeq z + \frac{X_i^2 + Y_i^2}{2z}\right)$$

$$\therefore\ \mu_{12} = \frac{J_{12}}{[J_{11} J_{22}]^{1/2}} = e^{j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2)}$$

$(\rho_i^2 = X_i^2 + Y_i^2)$

Since σ is monochromatic,

$J_{12}(P_1, P_2) = \langle u(P_1)\, u^*(P_2) \rangle = $ [illegible] and $I(x,y) = I_0 e^{-\frac{x^2+y^2}{W^2}} \Rightarrow A(x,y) = \sqrt{I_0}\, e^{-\frac{x^2+y^2}{2W^2}}$

Now $J_{12}(P_1,P_2) = I_0\, e^{-\frac{x_1^2+y_1^2+x_2^2+y_2^2}{2W^2}}$

$$J_{11}(P_1,P_1) = I_0\, e^{-\frac{x_1^2+y_1^2}{W^2}}$$

$$J_{22}(P_2,P_2) = I_0\, e^{-\frac{x_2^2+y_2^2}{W^2}}$$

$$\therefore \ \mu_{12} = \frac{J_{12}}{(J_{11}J_{22})^{1/2}} = 1$$

(c) From Gilbert-Zernike Theorem

$$\mu(x_1,y_1;x_2,y_2) = \frac{e^{-j\psi}\iint_{-\infty}^{\infty} I(\xi,\eta)\, e^{j\frac{2\pi}{\lambda z}(\Delta x\,\xi+\Delta y\,\eta)}\, d\xi\, d\eta}{\iint_{-\infty}^{\infty} I(\xi,\eta)\, d\xi\, d\eta}$$

here $\psi = \frac{\pi}{\lambda z}(\rho_2^2-\rho_1^2)$

$$I(\xi,\eta) = \begin{cases} I_0 & a_1 \le r = (\xi^2+\eta^2)^{1/2} \le a_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\iint_{-\infty}^{\infty} I(\xi,\eta)\,d\xi\,d\eta = \iint_A I_0\, d\xi\, d\eta = \pi I_0 (a_2^2 - a_1^2)$$

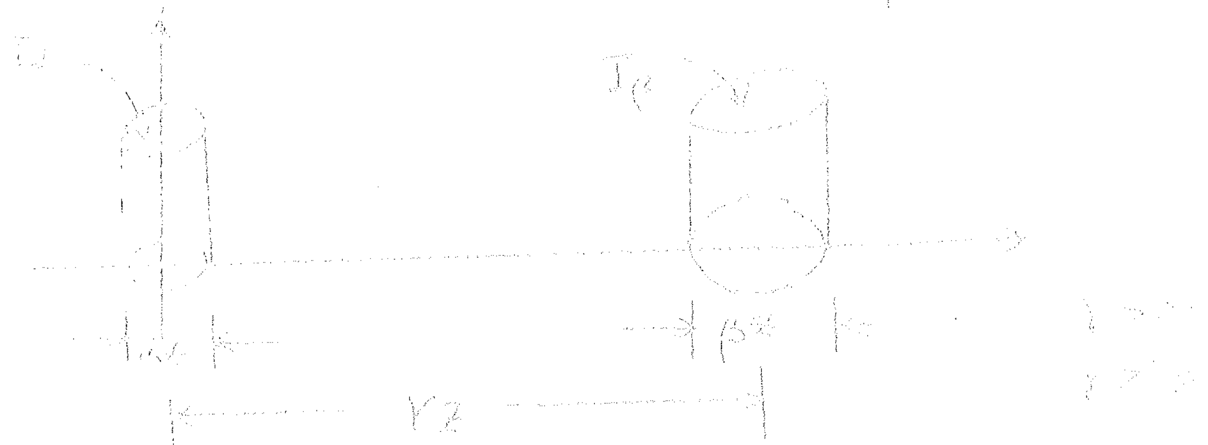
$$\iint_{-\infty}^{\infty} I(\xi,\eta)\, e^{j\frac{2\pi}{\lambda z}(\Delta x\,\xi+\Delta y\,\eta)}\, d\xi\, d\eta \qquad \left[\iint_{-\infty}^{\infty} = \iint_{A_2} - \iint_{A_1}\right]$$

$$= I_0\, \mathcal{F}\left\{\mathrm{circ}\left(\frac{r}{a_1}\right)\right\}\Big|_{\substack{\nu_x=\frac{\Delta x}{\lambda z}\\ \nu_y=\frac{\Delta y}{\lambda z}}} - I_0\, \mathcal{F}\left\{\mathrm{circ}\left(\frac{r}{a_2}\right)\right\}\Big|_{\substack{\nu_x=\frac{\Delta x}{\lambda z}\\ \nu_y=\frac{\Delta y}{\lambda z}}}$$

$$= I_0\left[\frac{a_2^2\, J_1\left(2\pi a_2\sqrt{\nu_x^2+\nu_y^2}\right)}{a_2\sqrt{\nu_x^2+\nu_y^2}} - \frac{a_1^2\, J_1\left(2\pi a_1\sqrt{\nu_x^2+\nu_y^2}\right)}{a_1\sqrt{\nu_x^2+\nu_y^2}}\right]\Big|_{\substack{\nu_x=\frac{\Delta x}{\lambda z}\\ \nu_y=\frac{\Delta y}{\lambda z}}}$$

[illegible]

$$I(x,y) = I_\alpha \, \mathrm{circ}\left(\frac{\sqrt{x^2+y^2}}{\alpha z}\right) + I_\beta \, \mathrm{circ}\left(\frac{\sqrt{(x-\gamma z)^2+y^2}}{\beta z}\right)$$



Using results, pp 136 ~ 140 of the notes

$$\mu(\rho) = e^{j\psi}\left[\frac{\alpha z \, I_\alpha J_1(\pi\alpha z\rho) + e^{j2\pi\gamma z V_x} \beta z \, I_\beta J_1(\pi\beta z\rho)}{\pi\rho z^2(\alpha^2 I_\alpha + \beta^2 I_\beta)}\right]$$

where $\rho = \sqrt{V_x^2 + V_y^2} = \sqrt{\left(\frac{\Delta x}{\lambda z}\right)^2 + \left(\frac{\Delta y}{\lambda z}\right)^2}$

One of the many possible solutions:

$\frac{I_\alpha}{I_\beta}$ can be determined by setting the [illegible]

of the interferometer so that [illegible] and measuring $|B_n|$ at that spacing, we find

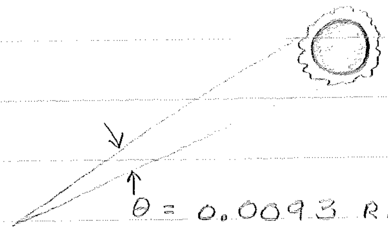
$$|B_n(s_0)| = \left| \frac{2J_1\left(\pi \frac{\beta}{\alpha} 1.22\right)}{1.22\,\pi \frac{\beta}{\alpha}\left(\frac{\alpha^2}{\beta^2}\frac{I_\alpha}{I_\beta} + 1\right)} \right|$$

$$\frac{I_\alpha}{I_\beta} = \left(\frac{\beta}{\alpha}\right)^2 \left\{ \frac{\left|2J_1\left(\pi \frac{\beta}{\alpha} 1.22\right)\right|}{1.22\,\pi \frac{\beta}{\alpha}\left|B_n\left(1.22\frac{\bar{\lambda}}{\alpha}\right)\right|} - 1 \right\}$$

This can be checked by measuring $|B_n|$ for a spacing $s = 1.22\frac{\bar{\lambda}}{\beta}$ and solving a similar equation with the variables α and β reversed. (May yield second solution.)

BOB MARKS
EE 5358
DUE 3/10/75
(SET # 4)

10

1. 

$\bar{\lambda} = 5500\ \text{Å}$

$\theta = 0.0093$ RAD

MODEL THE SUN AS A CIRCLE. FOR YOUNG'S EXPERIMENT, THE SPACING S FOR WHICH THE FRINGES WILL VANISH IS (FROM pg. 139):

$$S_0 = 1.22\ \frac{\bar{\lambda}}{\Theta}$$

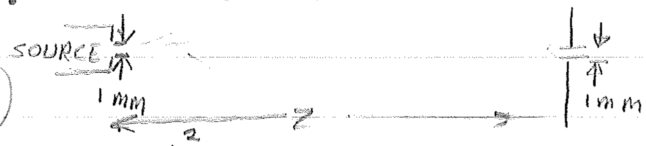
$$= 1.22\left(\frac{5500\ \text{Å}}{.0093}\right) = 7.22 \times 10^{5}\ \text{Å} = 7.22 \times 10^{-5}\ \text{m}$$

$$= 7.22 \times 10^{-2}\ \text{mm}$$

$$= 0.0722\ \text{mm}$$

10

2. $\lambda = 5000\,\text{Å}$



$A_S = (.5)^2 \pi \text{ mm}^2$

$A_c = (.5)^2 \pi \text{ mm}^2 \Rightarrow$ REQUIRED COHERENCE AREA @ z

$A_c A_S = (\bar{\lambda} z)^2$ $\Leftarrow$ Pg. 134

$z = \dfrac{\sqrt{A_c A_S}}{\bar{\lambda}}$

$= \dfrac{(.25)\pi \text{ mm}^2}{5000\,\text{Å}} \times \dfrac{1\,\text{Å}}{10^{-10}\text{ m}} \times \dfrac{1 \text{ m}}{10^3 \text{ mm}} = 1571 \text{ mm} = 1.57 \text{ m}$

3a. $A_c \triangleq \int_{-\infty}^{\infty}\int |\mu(\Delta X, \Delta Y)|^2 d\Delta X\, d\Delta Y$ (Pg. 134)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\int_{-\infty}^{\infty}\int I(\xi,\eta)\, e^{j\frac{2\pi}{\bar\lambda z}(\Delta X\xi + \Delta Y\eta)}\, d\xi\, d\eta}{\int_{-\infty}^{\infty}\int I(\xi,\eta)\, d\xi\, d\eta} \right|^2 d\Delta X\, d\Delta Y$$ (Pg 131)

10

LET $f_X = \frac{\Delta X}{\bar\lambda z}$; $f_Y = \frac{\Delta Y}{\bar\lambda z}$

$$\Rightarrow A_c = \frac{(\bar\lambda z)^2}{\left[\int_{-\infty}^{\infty}\int I(\xi,\eta)\, d\xi\, d\eta\right]^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty}\int I(\xi,\eta)\, e^{j2\pi(\xi f_X + \eta f_Y)}\, d\xi\, d\eta \right| df_X\, df_Y$$

$$= \frac{(\bar\lambda z)^2}{\left[\int_{-\infty}^{\infty}\int I(\xi,\eta)\, d\xi\, d\eta\right]^2} \int_{-\infty}^{\infty}\int \left| \mathcal{F}\{I(\xi,\eta)\} \right|^2 df_X\, df_Y$$

A FORM OF PARCEVAL'S THEM. IS

$$\int_{-\infty}^{\infty}\int |f(x)|^2 dx = \int_{-\infty}^{\infty}\int |\mathcal{F}\{f(x)\}|^2 df_x$$

THUS

$$A_c = (\bar\lambda z)^2 \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I^2(\xi,\eta)\, d\xi\, d\eta}{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(\xi,\eta)\, d\xi\, d\eta\right]^2}$$

b. $I(\xi,\eta) = I_0 P(\xi,\eta)$

$P(\xi,\eta)$ TAKES ON VALUES OF ONLY 1 & 0

$$\Rightarrow P^2(\xi,\eta) = P(\xi,\eta)$$

$$\Rightarrow A_c = (\bar\lambda z)^2 \frac{\int_{-\infty}^{\infty} I_0^2 P(\xi,\eta)\, d\xi\, d\eta}{\left[\int_{-\infty}^{\infty}\int I_0 P(\xi,\eta)\, d\xi\, d\eta\right]^2}$$

$$= \frac{(\bar\lambda z)^2}{\int_{-\infty}^{\infty}\int P(\xi,\eta)\, d\xi\, d\eta}$$

BUT $A_S = \int_{-\infty}^{\infty}\int P(\xi,\eta)\, d\xi\, d\eta$ IS THE TOTAL AREA OF THE SOURCE (ie AREA OVER WHICH $P(\xi,\eta) = 1$)

THUS

$$A_c = \frac{(\bar\lambda z)^2}{A_S}$$

10

4. UNDER QUASI-MONOCHROMATIC CONDITIONS, THE MUTUAL INTENSITY IN THE REAR FOCAL PLANE IS GIVEN IN SEC. 3.1.2 AS (EMPLOYING PARAXIAL APPROX):

$$J_f(x_1,y_1;x_2,y_2) = \frac{1}{(\lambda F)^2}\iiiint_{-\infty}^{\infty} J_0(\xi_1,\eta_1;\xi_2,\eta_2)\, e^{-j\frac{2\pi}{\lambda F}(x_1\xi_1+y_1\eta_1-x_2\xi_2-y_2\eta_2)}\, d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

WE MAY MODEL THE SOURCE AS IN SEC. 2.4.2:

$$J_0(\xi_1,\eta_1;\xi_2,\xi_2) = K\, I(\xi_1,\eta_1)\, \delta(\xi_1-\xi_2;\eta_1-\eta_2)$$

SUBSTITUTING GIVES

$$J_f(x_1,y_1;x_2,y_2) = \frac{K}{(\lambda F)^2}\iiiint_{-\infty}^{\infty} I(\xi_1,\eta_1)\, \delta(\xi_1-\xi_2;\eta_1-\eta_2)$$
$$\exp j\frac{2\pi}{\lambda F}(x_1\xi_1+y_1\eta_1-x_2\xi_2-y_2\eta_2)\, d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

$$= \frac{K}{(\lambda F)^2}\iint_{-\infty}^{\infty} I(\xi_1,\eta_1) \exp j\frac{2\pi}{\lambda F}(x_1\xi_1+y_1\eta_1-x_2\xi_1-y_2\eta_1)\, d\xi_1 d\eta_1$$

$$= \frac{K}{(\lambda F)^2}\iint_{-\infty}^{\infty} I(\xi,\eta)\, e^{j\frac{2\pi}{\lambda F}(\Delta x\xi+\Delta y\eta)}\, d\xi d\eta$$

WHERE $\Delta x = x_2 - x_1$ AND $\Delta y = y_2 - y_1$

IT FOLLOWS THAT

$$J_f(x_i,y_i;x_i,y_i) = \frac{K}{(\lambda F)^2}\iint_{-\infty}^{\infty} I(\xi,\eta)\, d\xi d\eta \quad ; \; i = 1,2$$

THUS

$$\mu(\Delta x,\Delta y) = \frac{J_f(x_1,y_1;x_2,y_2)}{[J_F(x_1,y_1;x_1,y_1)\, J_F(x_2,y_2;x_2,y_2)]^{1/2}}$$

$$= \frac{\iint_{-\infty}^{\infty} I(\xi,\eta)\, e^{j\frac{2\pi}{\lambda F}(\Delta x\xi+\Delta y\eta)}\, d\xi d\eta}{\iint_{-\infty}^{\infty} I(\xi,\eta)\, d\xi d\eta}$$

5a. FOR MONOCHROMATIC LIGHT (Pg. 5):

$$U(P_0,\nu) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1,\nu) \frac{1}{z} e^{j 2\pi r/\lambda} \chi(\theta)\, ds$$

10

THE POINT SOURCE GIVES (w $\chi(\theta)=1$)

$$U(P_0,\nu) = \frac{S}{j\lambda r} e^{j 2\pi r/\lambda}$$

$$\left[\begin{array}{l} \text{OR, SINCE } \quad u(P_0,t) = U(P_0,\nu) e^{-j 2\pi \nu t} \\ \underline{u}(P,t) = \frac{S}{j\lambda r} e^{-j 2\pi(\nu t - r/\lambda)} \end{array} \right]$$

EMPLOY THE PARAXIAL APPROXIMATION:

$$r \simeq z \qquad r \simeq z + \frac{x^2+y^2}{2z}$$

$$\Rightarrow U(P_0,\nu) = \frac{S}{j\lambda z} e^{j\frac{2\pi}{\lambda}\left(z + \frac{x^2+y^2}{2z}\right)} = \frac{S}{j\lambda z} e^{-j\frac{2\pi}{\lambda}\left(z + \frac{\rho^2}{2z}\right)}$$

NOW

$$\underline{J}_{12} = \langle u(P_1,t)\, u(P_2,t) \rangle$$

$$= U(P_1,\nu)\, U(P_2,\nu)$$

$$= \frac{S^2}{(\lambda z)^2} e^{j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2)}$$

$$\Rightarrow \underline{J}_{11} = \frac{I_0}{(\lambda z)^2} = J_{22} \Rightarrow \mu_{12} = \frac{J_{12}}{\sqrt{J_{11} J_{22}}} = e^{j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2)}$$

5b. AGAIN $(\chi_1 = \chi_2 = 1,\ r_1 = r_2 \simeq z)$

$$J_{12} = \frac{1}{(\lambda z)^2} \iiiint J'_{12}\, e^{-j\frac{2\pi}{\lambda}(r_2 - r_1)}\, ds_1\, ds_2$$

$$J'_{12} = \langle u_1(t)\, u_2(t)\rangle = I_0\, e^{-(x_1^2 + x_2^2 + y_1^2 + y_2^2)/2w^2}$$

SINCE $\quad u_i(t) = \sqrt{I_0'}\, e^{x^2 + y^2/2w}\quad e^{j(2\pi \nu_0 t - \phi)}$

APPLY PARAX:

$$r_2 - r_1 = (\rho_1^2 - \rho_2^2 + 2X\Delta\xi + 2Y\Delta\eta)/2z$$

THEN

$$J_{12} = \frac{I_0}{(\lambda z)^2} \iiiint e^{-(\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2)/2w^2}\, e^{-j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2 + 2\Delta\xi X + 2\Delta\eta Y)}\, ds_1\, ds_2$$

BUT $\quad \Delta\xi = \xi_2 - \xi_1 \quad ; \Delta\eta = \eta_2 - \eta_1$

$$J_{12} = \frac{I_0}{(\lambda z)^2}\, e^{-j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2)} \iint e^{-(\xi_1^2 + \eta_1^2)/2w^2}\, e^{+j\frac{2\pi}{\lambda f}(\xi_1 X + \eta_1 Y)}\, d\xi_1\, d\eta_1$$

$$\times \iint e^{-(\xi_2^2 + \eta_2^2)/2w^2}\, e^{-j\frac{2\pi}{\lambda f}(\xi_2 Y + \eta_2 Y)}\, d\xi_2\, d\eta_2$$

$$= \frac{I_0}{(\lambda z)^2}\, e^{-j\frac{\pi}{\lambda z}(\rho_1^2 - \rho_2^2)} \left[\int_{\infty}^{\infty}\!\!\int e^{-(\xi^2 + \eta^2)/2w^2} \cos\frac{2\pi}{\lambda f}(\xi Y + \eta Y)\, d\xi\, d\eta\right]^2$$

BUT

$$\int_{\infty}^{\infty} e^{-\xi^2/2w^2} \cos\frac{2\pi}{\lambda z}\xi X\, d\xi = \sqrt{2\pi}\, w\, e^{-2(\pi w X/\lambda z)^2}$$

$$\Rightarrow J_{12} = \frac{I_0}{(\lambda z)^2}\, e^{-j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}\, (2\pi)^2\, w^2\, e^{-\frac{4\pi^2}{(\lambda z)^2}(X^2 + Y^2) w^2}$$

$$J_{ii} = \frac{I_0}{(\lambda z)^2} \iiiint e^{(\xi_1^2 + \eta_1^2) 2w}\, ds_1\, ds_2 = (2\pi)^2\, w\, \frac{I_0}{(\lambda z)^2}$$

$$\Rightarrow \mu_{12} = e^{-j\frac{\pi}{\lambda f}(\rho_1^2 - \rho_2^2)}\, e^{-\left(\frac{2\pi w}{\lambda z}\right)^2 (X^2 + Y^2)}$$

Note soln.

c. $I(\xi) = I_0\left[\operatorname{circ}\frac{\sqrt{\xi^2+\eta^2}}{a_2} - \operatorname{circ}\frac{\sqrt{\xi^2+\eta^2}}{a_1}\right]$

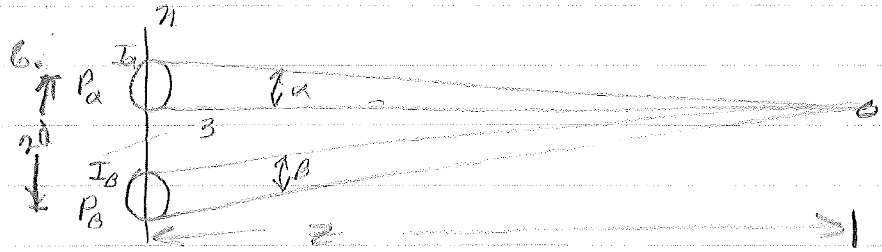
$$\mu_{12} = \frac{e^{-j\psi}\ \mathcal{F}[I(\xi,\eta)]\Big|_{\substack{\nu_x=\Delta x/\lambda z\\ \nu_y=\Delta y/\lambda z}}}{\int_{-\infty}^{\infty}\int I(\xi,\eta)\,d\xi\,d\eta}$$

$$\int_{-\infty}^{\infty}\int I(\xi,\eta)\,d\xi\,d\eta = I_0\pi(a_2^2-a_1^2)$$

$$\mathcal{F}[I(\xi,\eta)]\Big|_{\substack{\nu_y=\Delta y/\lambda z\\ \nu_x=\Delta x/\lambda z}} = I_0 a_2\ \frac{J_1\left(\frac{2\pi a_2}{\lambda z}\sqrt{\Delta x^2+\Delta y^2}\right)}{\lambda z\sqrt{\Delta x^2+\Delta y^2}} - I_0 a_1\ \frac{J_1\left(\frac{2\pi a_1}{\lambda z}\sqrt{\Delta x^2+\Delta y^2}\right)}{\lambda z\sqrt{\Delta x^2+\Delta y^2}}$$

$$\Rightarrow \mu_{12} = e^{-j\psi}\left[\frac{a_2 J_1\left(\frac{2\pi a_2}{\lambda z}\sqrt{\Delta x^2+\Delta y^2}\right) - a_1 J_1\left(\frac{2\pi a_1}{\lambda z}\sqrt{\Delta x^2+\Delta y^2}\right)}{\bar{\lambda} z\,(a_2^2-a_1^2)\sqrt{\Delta x^2+\Delta y^2}}\right]$$

6.



WE MAY APPROXIMATE THE STARS RADII BY:

$$a_\alpha = \frac{\alpha z}{2} \qquad a_\beta = \frac{\beta z}{2} \qquad (1)$$

THE SEPARATION, 2d, BETWEEN THE STARS IS

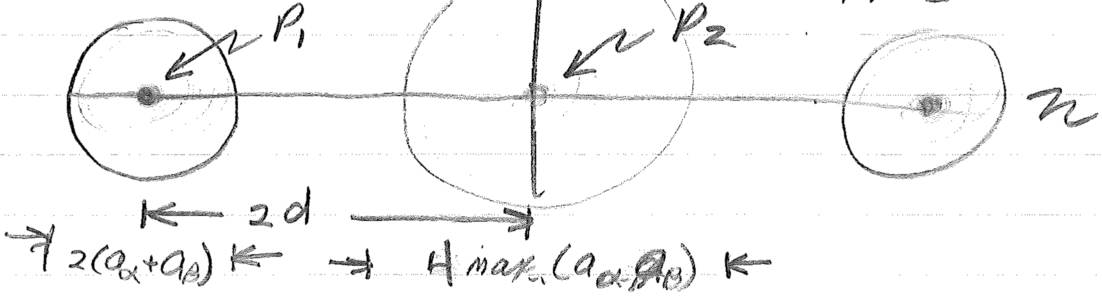
$$d = \frac{\gamma z}{2} \qquad (2)$$

THUS, CHOOSING APPROPRIATE COORDINATE

$$I(\xi,\eta) = I_\alpha \, \mathrm{circ} \frac{\sqrt{\xi^2+(\eta-d)^2}}{a_\alpha} + I_\beta \, \mathrm{circ} \frac{\sqrt{\xi^2+(\eta+d)^2}}{a_\beta}$$

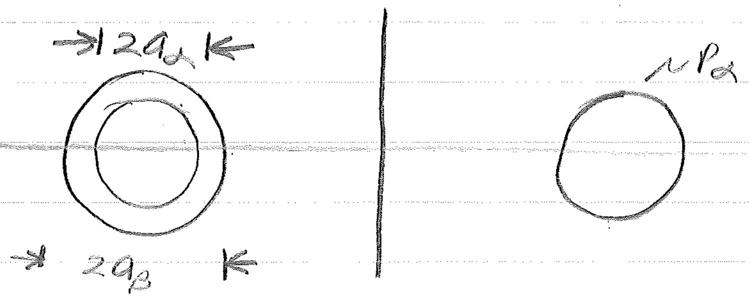
SUPPOSE WE USE INTERFEROMETER TO MEASSURE $|\mu_{12}(s)|$ AND PERFORM A FOURIER XFORM ON $|\mu_{12}(s)|^2$ WHICH GIVES THE AUTOCORRELATION OF $I(\xi,\eta)$, WHICH LOOKS LIKE: (SEE pg. 184)

FIG 1



SINCE $\gamma >> \alpha, \beta$, THESE 3 DISTRIBUTIONS WILL NOT OVERLAP.

VIEWING THE AUTOCORRELATION AS "SHIFTING", P_1 IS FORMED VIA



THUS, THE VALUE, C_{P_1} AT POINT P_1 IN FIG 1 IS

$$C_{P_1}^2 = I_\alpha I_\beta \pi \left(\min(a_\alpha, a_\beta)\right)^2 \qquad (3)$$

AT P_2, BOTH CIRCLES OVERLAP COMPLETELEY. THUS

$$C_{P_2} = \pi\left(I_\alpha^2 a_\alpha^2 + I_\beta^2 a_\beta^2\right) \qquad (4)$$

WE MAY MEASURE C_{P_1} AND C_{P_2} DIRECTLY. THUS Eq. 3 AND Eq. 4 (WITH Eqs 1 & 2) CONSTITUTE TWO EQUATIONS WITH TWO UNKNOWNS, FROM WHICH WE MAY COMPUTE I_α & I_β

(NOTE: ACTUALLY, THE $\mathcal{F}[|\mu_{12}|^2]$ IS PROPORTIONAL TO A SHIFTED VERSION OF $I * I$ (ALSO SCALED). THE PROPORTIONALITY CONSTANT, THOUGH, WILL BE ABSORBED IN THE C_p's)

EE 349 SPRING 1975

PROBLEM SET #5

DUE FRI 4/9

#1. Starting with the equation for the spectrum of image intensity for a partially coherent imaging system, as given on page 172 of the notes, show that:

(a) For incoherent object illumination, i.e. $\underline{J}_o(\Delta x, \Delta y) = K I_o \delta(\Delta x, \Delta y)$,

$$\frac{\mathcal{I}(\nu_x, \nu_Y)}{\mathcal{I}(0,0)} = \frac{T(\nu_x, \nu_Y)}{T(0,0)} \underline{\mathcal{H}}(\nu_x, \nu_Y)$$

where

$$\underline{T}(\nu_x, \nu_Y) \triangleq \iint_{-\infty}^{\infty} |\underline{t}_o(\xi, \eta)|^2 e^{-j2\pi(\nu_x \xi + \nu_Y \eta)} d\xi\, d\eta$$

and

$$\underline{\mathcal{H}}(\nu_x, \nu_Y) = \frac{\iint_{-\infty}^{\infty} \underline{P}(x_1, y_1)\underline{P}^*(x_1 - \lambda F \nu_x, y_1 - \lambda F \nu_Y)\, dx_1\, dy_1}{\iint_{-\infty}^{\infty} |\underline{P}(x_1, y_1)|^2\, dx_1\, dy_1}$$

(b) For fully coherent object illumination i.e. $J_o(\Delta x, \Delta y) = I_o$,

$$\mathscr{I}(\nu_x, \nu_y) = \frac{I_o}{(\lambda F)^2} \iint_{-\infty}^{\infty} \underline{S}\left(\frac{x_1}{\lambda F}, \frac{y_1}{\lambda F}\right) \underline{P}(x_1, y_1)$$

$$\cdot\ \underline{S}^*\left(\frac{x_1 - \lambda F \nu_x}{\lambda F}, \frac{y_1 - \lambda F \nu_y}{\lambda F}\right) \underline{P}^*(x_1 - \lambda F \nu_x, y_1 - \lambda F \nu_y)\, dx_1\, dy_1$$

where

$$\underline{S}(\nu_x, \nu_y) \triangleq \iint_{-\infty}^{\infty} t_o(\xi, \eta)\, e^{-j2\pi(\nu_x \xi + \nu_y \eta)}\, d\xi\, d\eta$$

3. An incoherent imaging system contains a gaussian random phase screen in its pupil. The normalized autocorrelation function of its random phase is

$$\bar{R}_\phi(\Delta x, \Delta y) = \exp\left[-\Delta^2/\Delta_0^2\right] \qquad 205$$

where $\Delta = \sqrt{\Delta x^2 + \Delta y^2}$

(a) Using the approximation $\ln(1+x) \cong x$ for $x \ll 1$, show that for large $\overline{\phi^2}$ the factor $\overline{\mathcal{H}}_t$ drops to $1/e$ at frequency

$$\nu_0 \cong \frac{\ell_0}{\bar{\lambda} F \sqrt{\overline{\phi^2}}}.$$

(b) Assuming $\ell_0 \cong 2\,mm$, find the spatial frequency at which the factor $\overline{\mathcal{H}}_t$ drops to $1/e$ for the specific numbers $\bar{\lambda} = 0.5 \times 10^{-4}\,cm$, $F = 10\,cm$ and for phase variances $\overline{\phi^2}$ of 10, 5 and 1 (radian)^2.

#3. A purely absorbing random screen is placed in the pupil of an incoherent imaging system. Prove that at all frequencies corresponding to spacings that are much larger than the correlation distance of the screen, the MTF (modulus of the OTF) is never smaller than $|\underline{t}_0 \underline{\mathcal{H}}_0|$, where $\underline{t}_0$ is the mean amplitude transmittance of the screen and $\underline{\mathcal{H}}_0$ is the OTF without the screen present. Is this conclusion necessarily valid for lower spatial frequencies?

(Hint: Remember the constraint $t \le 1$).

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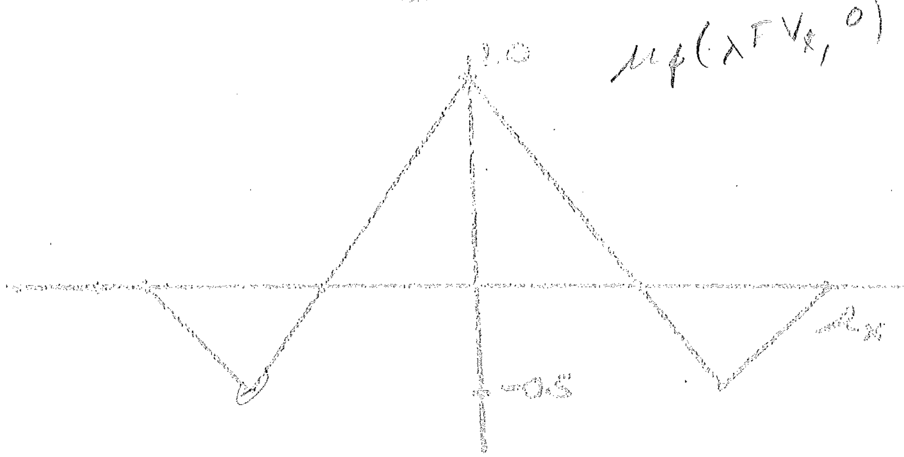
#4. X A gaussian, stationary random phase screen is placed in the pupil of an incoherent imaging system. The phase variance is 1 $=\sigma_\phi$ $(\text{radian})^2$ and the normalized autocorrelation function of phase is sketched below for a cut along the Δx axis.



Considering only spatial frequencies in the ν_x direction, what is the minimum (over ν_x) value of the ensemble average transfer function $\bar{\mu}_t(\nu_x, 0)$ due to the screen?

Problem Set #6 (Cont.)

#5. The angular spatial frequency Ω at which the long-exposure OTF falls to a value $1/e$ is directly proportional to $\sqrt[5]{\bar{\lambda}}$. If $\Omega_{1/e}$ is known to be 305 cycles per milliradian at $\bar{\lambda} = 5 \times 10^{-7}$ m, predict $\Omega_{1/e}$ for
$\bar{\lambda} = 0.488 \times 10^{-6}$ m (argon laser),
$\bar{\lambda} = 0.694 \times 10^{-6}$ m (ruby laser),
$\bar{\lambda} = 1.06 \times 10^{-6}$ m (Nd-glass laser),
and $\bar{\lambda} = 10.6 \times 10^{-6}$ m (CO_2 laser).

#6. An imaging system operating with $\bar{\lambda} = 5 \times 10^{-7}$ m must image through an adjacent region of turbulence (thickness z) which is homogeneous and isotropic and characterized by a strength parameter $C_n^2 = 10^{-15}\ m^{-2/3}$. Calculate the

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angular spatial frequency $\Omega_{1/e}$ at which $\bar{\mu}_A$ falls to 1/e for $z = 100\,m$, $z = 1000\,m$, $z = 10{,}000\,m$.

#7. A set of experimental measurements performed using an astronomical telescope and a star shows that $\bar{\mu}_A(\bar{\lambda}\Omega)$ falls to 1/e at a spatial frequency of 0.5 cycles per arc-second. The measurements were made with light of mean wavelength $\bar{\lambda} = 5 \times 10^{-7}\,m$. Find the value of $\int_0^\infty C_N^2(z)\,dz$ (256 BOT, 10^{-13}) looking up through the atmosphere. If we imagine the atmosphere to be a uniformly turbulent medium with $C_N^2 = 10^{-15}\,m^{-2/3}$ (246), what would the effective thickness or height of the turbulent atmosphere be? $10^5\,m$

OMIT

#8. Experiments show that the maximum angular resolution achievable under average atmospheric seeing conditions at $\lambda = 5 \times 10^{-7}$ m is about 0.5 cycles per arc second (1/e frequency).

(a) Using the results of the Kolmogorov theory, predict the maximum resolution (cycles per arc second) achievable at $\lambda = 10^{-1}$ m (microwaves).

(b) From the result of (a), estimate the maximum spacing (in meters) that can be used by a microwave interferometer without losing the phase of the complex coherence factor due to atmospheric turbulence effects.

(c) In fact, atmospheric effects do not in practice pose a significant obstacle at microwave frequencies. List all the reasons you can think of why our predictions in parts (a) and (b) are incorrect.

Marks

HW #5 Solution

(1) a) For incoherent illumination:

$J_o(\Delta x, \Delta y) = K I_o \delta(\Delta x, \Delta y)$

From what given in P. 170.

$$\mathcal{g}(\nu_x, \nu_y) = \iint_{-\infty}^{\infty} J_p'(x, y; x-\bar{\lambda}F\nu_x, y-\bar{\lambda}F\nu_y)\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} \underline{P}(x,y)\underline{P}^*(x-\bar{\lambda}F\nu_x, y-\bar{\lambda}F\nu_y) J_p(x,y; x-\bar{\lambda}F\nu_x, y-\bar{\lambda}F\nu_y)\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} \underline{P}(x,y)\underline{P}^*(x-\bar{\lambda}F\nu_x, y-\bar{\lambda}F\nu_y)\frac{K_o}{(\bar{\lambda}F)^2}\cdot \iint_{-\infty}^{\infty} J_o(\Delta\xi, \Delta\eta)\,\cdot$$

$$T(\Delta\xi, \Delta\eta; \bar{\lambda}F\nu_x, \bar{\lambda}F\nu_y)\cdot \exp\left\{-j\frac{2\pi}{\bar{\lambda}F}\left[(x-\bar{\lambda}F\nu_x)\Delta\xi\right.\right.$$

$$\left.\left. + (y-\bar{\lambda}F\nu_y)\Delta\eta\right]\right\} d\Delta\xi\, d\Delta\eta\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} \underline{P}(x,y)\underline{P}^*(x-\bar{\lambda}F\nu_x, y-\bar{\lambda}F\nu_y)\frac{K_o}{(\bar{\lambda}F)^2} K I_o T(0,0; \bar{\lambda}F\nu_x, \bar{\lambda}F\nu_y)\, dx\, dy$$

$$\mathcal{g}(0,0) = \iint_{-\infty}^{\infty} \underline{P}(x,y)\underline{P}^*(x,y)\frac{K_o}{(\bar{\lambda}F)^2} K I_o T(0,0;0,0)\, dx\, dy$$

and, in fact, $\tau(\nu_x, \nu_y) = K_o^{-1} T(0,0; \nu_x, \nu_y)$

$\tau(0,0) = K_o^{-1} T(0,0;0,0)$

Thus we get

$$\frac{\mathcal{g}(\nu_x, \nu_y)}{\mathcal{g}(0,0)} = \frac{\tau(\nu_x,\nu_y)}{\tau(0,0)} \frac{\iint \underline{P}(x,y)\underline{P}^*(x-\bar{\lambda}F\nu_x, y-\bar{\lambda}F\nu_y)\, dx\, dy}{\iint |\underline{P}(x,y)|^2\, dx\, dy} = \frac{\tau(\nu_x,\nu_y)}{\tau(0,0)} \mathcal{H}(\nu_x,\nu_y)$$

(b) For fully coherent case,

$$J_0(\Delta x, \Delta y) = I$$

Then $$J_p(x_1, y_1; x_2, y_2) = \frac{K_0 I_0}{(\lambda F)^2} \iint_{-\infty}^{\infty} T(\Delta\xi, \Delta\eta; \Delta x, \Delta y) \exp\left\{ j \frac{2\pi}{\lambda F} (x_2 \Delta\xi + y_2 \Delta\eta) \right\} d\Delta\xi\, d\Delta\eta$$

$$= \frac{I_0}{(\lambda F)^2} \iiiint_{-\infty}^{\infty} t_0(\xi_1, \eta_1) t_0^*(\xi_1 - \Delta\xi, \eta_1 - \Delta\eta) \exp\left\{ -j \frac{2\pi}{\lambda F} (x_1 \xi_1 + y_1 \eta_1) \right\}$$

$$\exp\left\{ j \frac{2\pi}{\lambda F} [x_2(\xi_1 - \Delta\xi) + y_2(\eta_1 - \Delta\eta)] \right\}$$

$$d\xi_1\, d\eta_1\, d(\xi_1 - \Delta\xi)\, d(\eta_1 - \Delta\eta)$$

$$= \frac{I_0}{(\lambda F)^2} \underline{S}\left(\frac{x_1}{\lambda F}, \frac{y_1}{\lambda F}\right) \underline{S}^*\left(\frac{x_2}{\lambda F}, \frac{y_2}{\lambda F}\right)$$

Now $$\mathcal{J}(\nu_x, \nu_y) = \iint_{-\infty}^{\infty} J_f'(x_1, y_1; x_1 - \lambda F \nu_x, y_1 - \lambda F \nu_y)\, dx_1\, dy_1$$

$$= \frac{I_0}{(\lambda F)^2} \iint_{-\infty}^{\infty} \underline{P}(x_1, y_1)\, \underline{S}\left(\frac{x_1}{\lambda F}, \frac{y_1}{\lambda F}\right) \underline{P}^*(x_1 - \lambda F \nu_x, y_1 - \lambda F \nu_y)$$

$$\cdot \underline{S}^*\left(\frac{x_1 - \lambda F \nu_x}{\lambda F}, \frac{y_1 - \lambda F \nu_y}{\lambda F}\right) dx_1\, dy_1$$

(2) Random phase screen, $\bar{\mu}_\phi(s_x, s_y) = \exp\left[-\frac{s^2}{s_0^2}\right]$

$$\bar{\mu}_t = \exp\left[-\overline{\phi^2}\,(1 - \bar{\mu}_\phi)\right]$$

(a) The factor $\bar{\mu}_t$ drops to $1/e$ when

$$\overline{\phi^2}\,(1 - \bar{\mu}_\phi) = \overline{\phi^2}\left(1 - \exp\left(-\frac{s^2}{s_0^2}\right)\right) = 1$$

$$\exp\left[-\frac{s^2}{s_0^2}\right] = 1 - \frac{1}{\overline{\phi^2}}$$

$$s^2 = s_0^2 \ln\left(\frac{\overline{\phi^2}}{\overline{\phi^2}-1}\right) = s_0^2 \ln\left(1+\frac{1}{\overline{\phi^2}-1}\right) \simeq s_0^2 \frac{1}{\overline{\phi^2}-1} \doteq \frac{s_0^2}{\overline{\phi^2}}$$

☆

for $\overline{\phi^2}$ large.

hence $\nu_0 = \dfrac{s}{\lambda F} = \dfrac{s_0}{\lambda F\sqrt{\overline{\phi^2}}}$

(b) For $\overline{\phi^2}$ large

since $s_0 = 2\,mm$, $\lambda = 5.0\times10^{-4}\,mm$, $f = 1.0\times10^{2}\,mm$

$\nu_0 = \dfrac{40}{\sqrt{\overline{\phi^2}}}$ cycles/mm

$\overline{\phi^2} = 10\,(rad)^2$, $\nu_0 = 12.6$ cyc/mm

$\overline{\phi^2} = 5\,(rad)^2$, $\nu_0 \doteq 17.9$ cyc/mm

} we can use the above approx.

For $\overline{\phi^2} = 1\,rad^2$, this approx. is no longer valid,

In fact, $\nu_0 \to \infty$ for $\overline{\phi^2} = 1$ rad. (∵ ☆)

(3) Random absorbing screen: $t_s(x,y) = t_0 + r(x,y)$

For spacings much larger than the correlation distance of the screen, $\mu_r \to 0$

and we have $\overline{H} = \dfrac{t_0^2}{t_0^2 + \overline{r^2}} H_0$

To minimize the factor $\dfrac{t_0^2}{t_0^2+\overline{r^2}}$, we maximize $\overline{r^2}$

We know $\bar{r}=0$, and $0 \le t_s \le 1$. $\Rightarrow -t_0 \le r \le 1-t_0$

hence we are trying to maximize $\int_{-t_0}^{1-t_0} r^2 p(r)dr$

and it is maximized when the probability mass is concentrated at the two extreme points on the interval

$\therefore \begin{cases} p(-t_0)+p(1-t_0)=1 \\ -t_0\, p(-t_0)+(1-t_0)\,p(1-t_0)=0 \quad (\because \text{zero mean}) \end{cases}$

solving $p(-t_0)=(1-t_0)$, $p(1-t_0)=t_0$

$$\therefore \overline{r^2_{max}} = t_0^2(1-t_0)+(1-t_0)^2 t_0 = t_0(1-t_0)$$

$$\bar{H} \ge \frac{t_0^2}{t_0^2+t_0(1-t_0)} H_0 = t_0 H_0 \, . \quad Q.E.D.$$

Since $\mu_r \ne 0$, in general, for low freq case and it might be negative valued, hence the above statement is not valid for low freq case.

(4) $\bar{\mu}_t(\Delta x, 0) = \exp\{-\sigma_\phi^2[1-\bar{\mu}_\phi(\Delta x, 0)]\}$

$\sigma_\phi^2 = 1$. $\quad \bar{\mu}_\phi|_{min} = -\frac{1}{2}$

$\therefore \bar{\mu}_t(\Delta x, 0)\big|_{min} = e^{-3/2} = 0.223$ *

(a) long exposure [illegible] OTF

$$\bar{\mu}_t(\bar{\lambda}\Omega) = \exp\left\{-\frac{1}{2}\cdot 2.91\,\bar{k}^2 C_N^2 z\,(\bar{\lambda}\Omega)^{5/3}\right\}$$

$$\Rightarrow\ k'\,\bar{k}^2(\bar{\lambda}\,\Omega_{1/e})^{5/3} = k'(2\pi)^2(\bar{\lambda}\,\bar{\lambda}^{-\frac{6}{5}}\,\Omega_{1/e})^{5/3} = 1$$

$$\Rightarrow\ \Omega_{1/e} = k\,\bar{\lambda}^{\frac{1}{5}}$$

Since we know $\Omega_{1/e} = 305\ (\text{m rad})^{-1}$ at $\bar{\lambda} = 5\times10^{-7}\,\text{m}$

$$\Rightarrow\ k = \frac{305}{(5\times10^{-7})^{1/5}} = 5553$$

Then for $\bar{\lambda} = 0.488\times10^{-6}\,\text{m}$, (argon) $\Rightarrow \Omega_{1/e} = 303.5$ (m rad)

$\bar{\lambda} = 0.694\times10^{-6}\,\text{m}$ (ruby) $\Rightarrow \Omega_{1/e} = 325.7$ (m rad)

$\bar{\lambda} = 1.06\times10^{-6}\,\text{m}$ (Nd-glass) $\Rightarrow \Omega_{1/e} = 354.5$ (m rad)

$\bar{\lambda} = 10.6\times10^{-6}\,\text{m}$ (CO_2) $\Rightarrow \Omega_{1/e} = 561.8$ (m rad)

(b) $\bar{\lambda} = 5\times10^{-7}\,\text{m}$, $C_N^2 = 10^{-15}\,\text{m}^{-2/3}$

Now $\frac{1}{e} = \bar{\mu}_A(\lambda\Omega_{1/e}) = \exp\left\{-57.44\,\frac{C_N^2 z}{\bar{\lambda}^{1/3}}\,\Omega_{1/e}^{5/3}\right\}$

i.e. $$57.44\,\frac{C_N^2 z}{\bar{\lambda}^{1/3}}\,\Omega_{1/e}^{5/3} = 1$$

$$\Rightarrow\ \Omega_{1/e} = \bar{\lambda}^{1/5}(57.44\,C_N^2)^{-\frac{3}{5}}(z)^{-\frac{3}{5}}$$

z : (m)	$\Omega_{1/e}$ (cyc/rad)
100 m	3.05×10^{5}

1000 m $\qquad$ 7.66×10^{4}

10000 m $\qquad$ 1.92×10^{4}

(7) $\bar{\mu}_f(\bar{\lambda}\Omega_1) = \exp\left\{-\frac{1}{2} \cdot 2.91\, \bar{k}^2 \cdot \int_0^z C_n^2(z)\,dz\, (\bar{\lambda}\Omega_1)^{5/3}\right\}$

falls to $1/e$ when

$$\frac{2.91}{2}(2\pi)^2 \frac{\Omega_1^{5/3}}{\bar{\lambda}^{1/3}} \int_0^z C_N^2(z)\,dz = 1$$

$\bar{\lambda} = 5 \times 10^{-7}\,m$

$$\Omega_1 = 0.5\ \frac{\text{cycles}}{\text{arc sec}}\ \frac{3600\ \text{arc sec}}{1\ \text{deg.}}\ \frac{360\ \text{deg}}{2\pi\ \text{rad}} = 1.03 \times 10^{5}\ \text{cyc}/\text{rad}$$

$$\Rightarrow \int_0^z C_N^2(z)\,dz = \frac{2}{2.91\,(2\pi)^2}\ \frac{(5\times10^{-7}\,m)^{1/3}}{(1.03\times10^{5})^{5/3}} = 6.09 \times 10^{-13}\ m^{1/3}$$

for uniformly turbulent medium $C_N^2 = 10^{-15}\ m^{-2/3}$

$\Rightarrow z_{eff} = 609\ m.$

(8) (a) $\bar{\lambda} = 5 \times 10^{-7}\,m$ (visible) $\Omega_{1/e} = 0.5\ \frac{\text{cycles}}{\text{arc-sec}}$; $\Omega_1 \propto \bar{\lambda}^{1/5}$

hence for $\bar{\lambda} = 0.1\,m$ (microwave)

$$\Omega_{1/e} = 0.5\left(\frac{0.1}{5\times10^{-7}}\right)^{1/5} = 5.74\ \frac{\text{cycles}}{\text{arc-sec}}$$

b) $S_{max} = \lambda F \nu = \bar{\lambda}\,\Omega_{1/e}$

$$= (0.1\,m)\left(5.74\ \frac{\text{cycles}}{\text{arc-sec}}\right)\left(2.06\times10^{5}\ \frac{\text{arc sec}}{\text{rad}}\right) = 1.18\times10^{[illegible]}$$

c) The result obtained in part [illegible] [illegible]

a gross underestimate since it is based

on a theory which neglected saturation

of the phase structure function.

(i.e., in this case, Rytov approximation

is no longer good).

ROBERT J. MARKS II.

09 APR 1976

(1-1)

1a. THE EQUATION ON Pg. 172 SAYS

$$\mathcal{I}(V_x, V_y) = \iint_{-\infty}^{\infty} J_p'(x_1, Y_1; x_1 - \lambda F V_x, Y_1 - \lambda F V_y)\, dx_1\, dY_1 \quad ①$$

WHERE

$$V_x = \frac{\Delta x}{\lambda F} \; ; \quad V_y = \frac{\Delta Y}{\lambda F} \quad (2)$$

WE ARE GIVEN THAT

$$J_o(\Delta x, \Delta Y) = K I_o\, \delta(\Delta x, \Delta Y) \quad (3)$$

FROM Pg. 170,

$$J_p(x_1, Y_1; x_2, Y_2) = \frac{K_o}{(\lambda F)^2} \iint_{-\infty}^{\infty} J_o(\Delta\xi, \Delta\eta)\, T(\Delta\xi, \Delta\eta; \Delta x, \Delta Y)\, e^{-j\frac{2\pi}{\lambda F}(x_2\Delta\xi + Y_2\Delta\eta)}\, d\Delta\xi\, d\Delta\eta \quad (3b)$$

SUBSTITUTING (3):

$$J_p(x_1, Y_1; x_2, Y_2) = \frac{K K_o I_o}{(\lambda F)^2} \iint_{-\infty}^{\infty} \delta(\Delta\xi, \Delta\eta)\, T(\Delta\xi, \Delta\eta; \Delta x, \Delta Y)\, e^{-j\frac{2\pi}{\lambda F}(x_2\Delta\xi + Y_2\Delta\eta)}\, d\Delta\xi\, d\Delta\eta$$

$$= \frac{K K_o I_o}{(\lambda F)^2}\, T(0, 0; \Delta x, \Delta Y) \quad (4)$$

FROM Pg. 169

$$T(\Delta\xi, \Delta\eta; \Delta x, \Delta Y) \triangleq \frac{1}{K_o} \iint_{-\infty}^{\infty} t_o(\xi_1, \eta_1)\, t_o^*(\xi_1 - \Delta\xi, \eta_1 - \Delta\eta)\, e^{-j\frac{2\pi}{\lambda F}(\Delta x\xi_1 + \Delta Y\eta_1)}\, d\xi_1\, d\eta_1 \quad (5)$$

WHERE, FROM THE TOP OF 169:

$$K_o \triangleq \iint_{-\infty}^{\infty} |t(\xi, \eta)|^2\, d\xi\, d\eta \quad (6)$$

IT FOLLOWS THAT (6b)

$$T(0, 0; \Delta x, \Delta Y) = \frac{1}{K_o} \iint_{-\infty}^{\infty} t_o(\xi_1, \eta_1)\, t_o^*(\xi_1, \eta_1)\, e^{-j\frac{2\pi}{\lambda F}(\Delta x\xi_1 + \Delta Y\eta_1)}\, d\xi_1\, d\eta_1$$

$$= \frac{1}{K_o} \iint_{-\infty}^{\infty} |t_o(\xi, \eta)|^2\, e^{-j\frac{2\pi}{\lambda F}(\Delta x\xi + \Delta Y\eta)}\, d\xi\, d\eta$$

SUBSTITUTING INTO (4):

$$J_p(X_1, Y_1; X_2, Y_2) = \frac{K K_o I_o}{(\lambda F)^2} \frac{1}{K_o} \iint_{-\infty}^{\infty} |t_o(\xi, \eta)|^2\, e^{-j\frac{2\pi}{\lambda F}[\Delta x\xi + \Delta Y\eta]}\, d\xi\, d\eta$$

$$= \frac{K I_o}{(\lambda F)^2} \iint_{-\infty}^{\infty} |t_o(\xi, \eta)|^2\, e^{-j\frac{2\pi}{\lambda F}[\Delta x\xi + \Delta Y\eta]}\, d\xi\, d\eta \quad (7)$$

(CONT →)

FROM pg. 170

$$J_P'(X_1,Y_1;X_2,Y_2) = P(X_1,Y_1)\,P^*(X_2,Y_2)\,J_P(X_1,Y_1;X_2,Y_2) \quad (8)$$
$$= \frac{KI_o}{(\lambda F)^2} P(X_1,Y_1)\,P^*(X_2,Y_2) \iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 e^{-j\frac{2\pi}{\lambda F}(\Delta X\xi+\Delta Y\eta)}\,d\xi\,d\eta$$

BUT

$$\Delta X = X_1 - X_2 \quad ; \quad \Delta Y = Y_1 - Y_2 \quad (9)$$

THUS, WE MAY WRITE (8) AS

$$J_P'(X_1,Y_1;X_2,Y_2) = J_P'(X_1,Y_1;X_1-\Delta X,\,Y_1-\Delta Y)$$
$$= \frac{KI_o}{(\lambda F)^2} P(X_1,Y_1)\,P^*(X_1-\Delta X,\,Y_1-\Delta Y) \iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 e^{-j\frac{2\pi}{\lambda F}(\Delta X\xi+\Delta Y\eta)}\,d\xi\,d\eta$$

WHICH CAN BE REWRITTEN FROM (2) AS

$$J_P'(X_1,Y_1;X_1-\lambda F V_X,\,Y_1-\lambda F V_Y)$$
$$= \frac{KI_o}{(\lambda F)^2} P(X_1,Y_1)\,P^*(X_1-\lambda F V_X,\,Y_1-\lambda F V_Y)$$
$$\times \iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 e^{-j2\pi(\xi V_X+\eta V_Y)}\,d\xi\,d\eta$$

SUBSTITUTING INTO (1):

$$\mathcal{D}(V_X,V_Y) = \frac{KI_o}{(\lambda F)^2} \iint_{-\infty}^{\infty} P(X_1,Y_1)\,P^*(X_1-\lambda F V_X,\,Y_1-\lambda F V_Y)\,dX_1\,dY_1$$
$$\times \iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 e^{-j2\pi(\xi V_X+\eta V_Y)}\,d\xi\,d\eta \quad (10)$$

IT FOLLOWS THAT

$$\mathcal{D}(0,0) = \frac{KI_o}{(\lambda F)^2} \iint_{-\infty}^{\infty} |P(X_1,Y_1)|^2 dX_1\,dY_1 \iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 d\xi\,d\eta$$

AND THAT

$$\frac{\mathcal{D}(V_X,V_Y)}{\mathcal{D}(0,0)} = \frac{\iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 e^{-j2\pi(\xi V_X+\eta V_Y)}\,d\xi\,d\eta}{\iint |t_o(\xi,\eta)|^2 d\xi\,d\eta} \times \frac{\iint_{-\infty}^{\infty} P(X_1,Y_1)\,P^*(X_1-\lambda F V_X,\,Y_1-\lambda F V_Y)\,dX_1\,dY_1}{\iint_{-\infty}^{\infty} |P(X_1,Y_1)|^2 dX_1\,dY_1}$$

$$= \frac{\mathcal{G}(V_X,V_Y)}{\mathcal{G}(0,0)}\,\mathcal{H}(V_X,V_Y)$$

WHERE $\mathcal{G}(V_X,V_Y) = \iint_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 e^{-j2\pi(V_X\xi+V_Y\eta)}\,d\xi\,d\eta$

$$\Rightarrow \mathcal{G}(0,0) = \int_{-\infty}^{\infty} |t_o(\xi,\eta)|^2 d\xi\,d\eta = K_o$$

AND

$$\mathcal{H}(V_X,V_Y) = \frac{\iint_{-\infty}^{\infty} P(X_1,Y_1)\,P^*(X_1-\lambda F V_X,\,Y_1-\lambda F V_Y)\,dX_1\,dY_1}{\int_{-\infty}^{\infty} |P(X_1,Y_1)|^2 dX_1\,dY_1}$$

(CONT →)

b. FROM 172

$$\mathcal{O}(V_x,V_Y) = \iint_{-\infty}^{\infty} J_p'(x_1,Y_1;x_1-\lambda F V_x, Y_1-\lambda F V_x)\,dx_1\,dY_1 \quad (1)$$

BUT, FROM 170

$$J_p'(x_1,Y_1;X_2,Y_2) = P(x_1,Y_1)\,P^*(x_2,Y_2)\,J_p(x_1,Y_1;x_2,Y_2)$$

OR EQUIVALENTLY

$$J_p'(x_1,Y_1;x_1-\lambda F V_x, Y_1-\lambda F V_x) = P(x_1,Y_1)\,P^*(x_1-\lambda F V_x, Y_1-\lambda F V_Y)$$
$$\times J_p(x_1,Y_1;x_1-\lambda F V_x, Y_1-\lambda F V_Y)$$

PLUG INTO (1):

$$\mathcal{O}(V_x,V_Y) = \iint_{-\infty} P(x_1,Y_1)\,P^*(x_1-\lambda F V_x, Y_1-\lambda F V_Y)\,J_p(x_1,Y_1;x_1-\lambda F V_x, Y_1-\lambda F V_Y)\,dx_1\,dY_1 \quad (2)$$

BUT, FROM 169:

$$J_p(x_1,Y_1;X_2,Y_2) = \frac{1}{(\lambda F)^2}\iint_{-\infty}^{\infty} J_o(\Delta\xi,\Delta\eta)\iint_{-\infty}^{\infty} t_o(\xi_1,\eta_1)\,t_o^*(\xi_1-\Delta\xi,\eta_1-\Delta\eta)$$
$$e^{-j\frac{2\pi}{\lambda F}(\Delta x\,\xi_1+\Delta Y\,\eta_1)}\,d\xi_1\,d\eta_1\; e^{-j\frac{2\pi}{\lambda F}(x_2\Delta\xi+Y_2\Delta\eta)}\,d\Delta\xi\,d\Delta\eta \quad (3)$$

BUT, FOR FULLY COHERENT LIGHT: $J_o(\Delta x,\Delta Y) = I_o$

$$\Rightarrow J_p(x_1,Y_1;X_2,Y_2) = \frac{I_o}{(\lambda F)^2}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} t_o(\xi_1,\eta_1)\,t_o^*(\xi_1-\Delta\xi,\eta_1-\Delta\xi) \quad (4)$$
$$e^{-j\frac{2\pi}{\lambda F}(\Delta x\,\xi_1+\Delta Y\,\eta_1)}\,d\xi_1\,d\eta_1\; e^{-j\frac{2\pi}{\lambda F}(x_2\Delta\xi+Y_2\Delta\eta)}\,d\Delta\xi\,d\Delta\eta$$
$$= \frac{I_o}{\lambda F^2}\iint_{-\infty}\iint_{-\infty} t_o(\xi_1,\eta_1)\,t_o^*(\xi_2,\eta_2)\,e^{-j\frac{2\pi}{\lambda F}(\Delta x\,\xi_1+\Delta Y\,\eta_1)}\,d\xi_1\,d\eta_1$$
$$e^{-j\frac{2\pi}{\lambda F}(x_2\Delta\xi+Y_2\Delta\eta)}\,d\Delta\xi\,d\Delta\eta$$
$$= \frac{I_o}{(\lambda F)^2}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} t_o(\xi_1,\eta_1)\,e^{-j\frac{2\pi}{\lambda F}(\Delta x\,\xi_1+\Delta Y\,\eta_1)}\,d\xi_1\,d\eta_1$$
$$t_o^*(\xi_2,\eta_2)\,e^{-j\frac{2\pi}{\lambda F}(x_2\Delta\xi+Y_2\Delta\eta)}\,d\Delta\xi\,d\Delta\eta$$

BUT $V_x = \dfrac{\Delta X}{\lambda F}$ $\quad V_Y = \dfrac{\Delta Y}{\lambda F}$

$$\Rightarrow J_p(x_1,Y_1;X_2,Y_2) = \frac{I_o}{(\lambda F)^2}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} t_o(\xi_1,\eta_1)\,e^{-j2\pi(\xi_1 V_x+\xi_2 V_Y)}\,d\xi_1\,d\eta_1$$
$$t_o^*(\xi_1-\Delta\xi,\eta_2-\Delta\eta)\,e^{-j\frac{2\pi}{\lambda F}(X_2\Delta\xi+Y_2\Delta\eta)}\,d\Delta\xi\,d\Delta Y$$
$$= \frac{I_o}{(\lambda F)^2}\iint_{-\infty}^{\infty} t_o(\xi_1,\eta_1)\,e^{-j2\pi(V_x\xi+V_Y\eta)}\,d\xi\,d\eta \quad (5)$$
$$\times\iint_{-\infty}^{\infty} t_o^*(\xi_1-\lambda F V_x,\eta_1-\lambda F V_x)\,e^{-j2\pi(X_2V_x+Y_2V_Y)}\,dV_x\,dV_Y$$

(CONT →)

DEFINE

$$S\left(\frac{x_1}{\lambda F}, \frac{y_1}{\lambda F}\right) \triangleq \int_{-\infty}^{\infty}\!\!\int t_o(\xi,\eta)\, e^{-j2\pi(\nu_x \xi + \nu_y \eta)}\, d\xi\, d\eta$$

AND (5) BECOMES

$$\gamma_p(x_1, y_1; x_2 = \lambda F \nu_x, y_2 = \lambda F \nu_y) = \frac{I_o}{(\lambda F)^2} S\left(\frac{x_1}{\lambda F}, \frac{y_1}{\lambda F}\right) S^*\left(\frac{x_1 - \lambda F \nu_x}{\lambda F}, \frac{y_1 - \lambda F \nu_y}{\lambda F}\right)$$

PLUG INTO (2) GIVES DESIRED ANSWER:

$$\mathcal{I}(\nu_x, \nu_y) = \frac{I_o}{(\lambda F)^2} \int_{-\infty}^{\infty}\!\!\int S\left(\frac{x_1}{\lambda F}, \frac{y_1}{\lambda F}\right) P(x_1, y_1) S^*\left(\frac{x_1 - \lambda F \nu_x}{\lambda F}, \frac{y_1 - \lambda F \nu_x}{\lambda F}\right) P^*(x_1 - \lambda F \nu_x, y_1 - \lambda F \nu_y)\, dx_1\, dy_1$$

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2a. FROM Pg. 205:

$$\underline{\mu}_t(\lambda F v_x, \lambda F v_y) = \exp\left[-\sigma_\phi^2 \{1 - \underline{\mu}_\phi(\lambda F v_x, \lambda F v_y)\}\right]$$

WE ARE GIVEN THAT

$$\underline{\mu}_\phi(s_x, s_y) = \exp\left[-s^2/s_0^2\right] \; ; \; s^2 = s_x^2 + s_y^2$$

IT FOLLOWS THAT, FOR $s^2 = (\lambda F)^2(v_x^2 + v_y^2)$

$$\underline{\mu}_t(s) = \exp\left[-\sigma_\phi^2\{1 - e^{-s^2/s_0^2}\}\right]$$

TAKE THE NAPERIAN LOG OF BOTH SIDES:

$$\ln \underline{\mu}_t(s) = -\sigma_\phi^2\{1 - e^{-s^2/s_0^2}\}$$

IT FOLLOWS THAT $\underline{\mu}_t(s)$ WILL FALL TO $\frac{1}{e}$ OF ITS PEAK VALUE WHEN $\ln \underline{\mu}_t(s) = -1$. THUS

$$+1 = +\sigma_\phi^2\{1 - e^{-s^2/s_0^2}\}$$

$$1/\sigma_\phi^2 = 1 - e^{-s^2/s_0^2}$$

OR

$$e^{-s^2/s_0^2} = 1 - \frac{1}{\sigma_\phi^2}$$

AGAIN, TAKE THE NAPERIAN LOG

$$-\frac{s^2}{s_0^2} = \ln(1 - 1/\sigma_\phi^2)$$

FOR $\overline{\phi^2}$ LARGE, WE HAVE σ_ϕ^2 LARGE AND $\frac{1}{\sigma_\phi^2} \ll 1$. THUS, WE MAY APPROXIMATE

$$\ln(1 - 1/\sigma_\phi^2) \simeq -1/\sigma_\phi^2 = -1/\overline{\phi^2}$$

THUS

$$\frac{s^2}{s_0^2} \simeq +1/\overline{\phi^2}$$

$$s^2 \simeq \frac{s_0^2}{\overline{\phi^2}}$$

$$(\lambda F v_0)^2 = s_0^2/\overline{\phi^2}$$

WHERE, DUE TO SYMMETRY, WE LET $s = \lambda F v_r$

THUS

$$v_0 = \frac{s_0}{\lambda F \sqrt{\overline{\phi^2}}}$$

(CONT →)

b. $s_0 = 2$ mm

$\bar{\lambda} = 0.5 \times 10^{-4}$ cm

$F = 10$ cm

- $\overline{\phi^2} = 10$

$$\Rightarrow \nu_0 = \frac{2 \text{ mm}}{(0.5 \times 10^{-4} \text{cm})(10 \text{ cm})} \left(\frac{1 \text{ cm}}{10 \text{ mm}}\right)^2 \frac{1}{\sqrt{10}}$$

$$= \frac{40}{\sqrt{10}\,\text{mm}}$$

$$= 12.65 \ \frac{\text{LINES}}{\text{mm}}$$

- $\overline{\phi^2} = 5$

$$\Rightarrow V_0 = \frac{40}{\sqrt{5}\,\text{mm}}$$

$$= 17.89 \ \frac{\text{LINES}}{\text{mm}}$$

- $\overline{\phi^2} = 1$

$$\Rightarrow V_0 = \frac{40}{\sqrt{1}\,\text{mm}}$$

$$= 40 \ \frac{\text{LINES}}{\text{mm}}$$

(actually for $\overline{\phi^2} = 1$, approx no longer valid and $\nu_0 \rightarrow \infty$).

3. FOR THIS, IT SEEMS LIKE THE EQ. @ THE BOTTOM OF 202 LOOKS APPLICABLE:

$$\mathcal{H}(\nu_X, \nu_Y) = \frac{t_0^2}{t_0^2 + \sigma_r^2} \mathcal{H}_0(\nu_X, \nu_Y) + \frac{\sigma_r^2}{t_0^2 + \sigma_r^2} \mathcal{H}_0(\nu_X, \nu_Y) \mu_r(\lambda F \nu_X, \lambda F \nu_Y)$$

IF WE GREATLY EXCEED THE CORRELATION DISTANCE, THEN $\mu_r(\lambda F \nu_X, \lambda F \nu_Y) \approx 0$ AND WE ARE LEFT WITH

$$\mathcal{H}(\nu_X, \nu_Y) = \frac{t_0^2}{t_0^2 + \sigma_r^2} \mathcal{H}_0(\nu_X, \nu_Y)$$

SINCE WE ARE DEALING WITH A PURELY ABSORBING SCREEN, t_0 IS REAL ∴

$$|\mathcal{H}(\nu_X, \nu_Y)| = \frac{t_0^2}{t_0^2 + \sigma_r^2} |\mathcal{H}_0(\nu_X, \nu_Y)|$$

SUPPOSEDLY, THIS COULD BE SIMPLIFIED BY USING INEQUALITIES LIKE $\sigma_r^2 + t_0^2 \leq 1$ AND $t_0^2 < t_0$

BUT...

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4. WE KNOW FROM Pg 205, THAT (IN ONE DIMENSION):

$$\mu_{u}(\Delta_x) = \exp\left[-\sigma_\phi^2 \{1 - \mu_\phi(\Delta_x)\}\right]$$

WHERE $\Delta_x = \lambda f \nu_x$

THE PHASE VARIANCE, σ_ϕ^2, IS GIVEN AS UNITY.

THUS

$$\mu_{u}(\Delta_x) = e^{-(1-\mu_\phi(\Delta_x))} = e^{\mu_\phi(\Delta_x)-1}$$

NOW, FOR $g(t)$ REAL, IT FOLLOWS FROM THE STRICTLY MONOTONIC NATURE OF e^x WILL YIELD THE SAME EXTREMA, (WITH EQUIVALENT POLARITY) OF $g(t)$ AND $e^{g(t)}$, IT FOLLOWS THAT

$\mu_u(\Delta_x) = e^{\mu_\phi(\Delta_x)-1}$ WILL BE MINIMUM WHEN $\mu_\phi(\Delta_x) - 1$, OR SIMPLY $\mu_\phi(\Delta_x)$ IS MINUMUM. FROM THE SCETCH, THE MINIMUM VALUE OF $\mu_\phi(\Delta_x)$ IS -0.5.

THUS, THE MINIMUM VALUE OF $\mu_u(\Delta_x)$ IS

$$\mu_{u_{MIN}} = e^{(-0.5)-1}$$
$$= e^{-1.5} = 0.22313016\ldots$$

5. $\Omega_{\frac{1}{e}} \sim \sqrt[5]{\bar{\lambda}}$

10

WE KNOW THAT $\Omega_{\frac{1}{e}} = 305 \frac{\text{CYCLES}}{\text{MILIRAD}}$ CORRESPONDS TO $\bar{\lambda} = 5 \times 10^{-7}$ m. WE WRITE THE PROPORTIONALITY AS

$$\Omega_{\frac{1}{e}} = C \sqrt[5]{\bar{\lambda}}$$

IT FOLLOWS THAT

$$C = \Omega_{1/e} / \sqrt[5]{\bar{\lambda}} = \frac{305}{\sqrt[5]{5 \times 10^{-7}}} \frac{\text{CYCLES}}{\text{MILIRAD} \cdot m^{1/5}}$$

$$= 5.55 \times 10^{3} \frac{\text{CYCLES}}{\text{MILIRAD} \cdot m^{1/5}}$$

THUS, WITH $\Omega_{\frac{1}{e}}$ IN $\frac{\text{CYCLES}}{\text{MILIRADIAN}}$ AND λ IN METERS, WE HAVE

$$\Omega_{\frac{1}{e}} = C \sqrt[5]{\bar{\lambda}}$$

ENT ⇐ A LITTLE H/P 25 PROGRAM

5

$\frac{1}{x}$

Y^x

RCL 7 (C)

X

GTO 00

$\bar{\lambda} = 0.488 \times 10^{-6}$ m $\Rightarrow \Omega_{\frac{1}{e}} = 303.5 \frac{\text{CYCLES}}{\text{MILIRADIAN}}$

$\bar{\lambda} = 0.694 \times 10^{-6}$ m $\Rightarrow \Omega_{\frac{1}{e}} = 325.7$ "

$\bar{\lambda} = 1.06 \times 10^{-6}$ m $\Rightarrow \Omega_{\frac{1}{e}} = 354.5$ "

$\bar{\lambda} = 10.6 \times 10^{-6}$ $\Rightarrow \Omega_{\frac{1}{e}} = 561.8$ "

6. FROM THE BOTTOM OF 246:

$$\Omega_{\frac{1}{e}} = \frac{\bar{\lambda}^{\frac{1}{5}}}{(57.44\, C_N^2 Z)^{3/5}} \quad \left(\frac{\text{CYCLES}}{\text{RADIAN}}\right)$$

THIS IS OBVIOUSLY A JOB FOR THE H/P 25:

$C_N^2 = 10^{-15}\, m^{-2/3} \xrightarrow{STO} 7$

$\bar{\lambda} = 5 \times 10^{-7} \xrightarrow{STO} 6$

$57.44 \xrightarrow{STO} 5$

RCL 7	$\frac{1}{x}$
×	RCL 6
RCL 5	ENT
×	5
ENT	$\frac{1}{x}$
3	Y^x
ENT	×
5	GTO 00
÷	
Y^x	

$Z = 100\ m \Rightarrow \Omega_{\frac{1}{e}} = 3.05 \times 10^5$ $\frac{\text{CYCLES}}{\text{RAD}}$ = 305 $\frac{\text{CYCL}}{\text{m RAD}}$

$Z = 1000\ m \Rightarrow \Omega_{\frac{1}{e}} = 7.66 \times 10^4$ " = 76.6 "

$Z = 10{,}000\ m \Rightarrow \Omega_{\frac{1}{e}} = 1.92 \times 10^4$ " = 19.2 "

7. FROM THE BOTTOM OF 2.5.6:

$$\mu_H(\lambda\Omega) = \exp\left\{-\frac{1}{2}\, 2.91\, k^2 \int_0^z C_N^2(z)\,dz\; (\lambda\Omega)^{5/3}\right\} \quad (1)$$

WE ARE GIVEN

$$\Omega_{\frac{1}{e}} = \frac{1}{2}\ \frac{\text{CYCLE}}{\text{ARCSEC}}$$

$$= \frac{1}{2}\ \frac{\text{CYCLE}}{\text{ARCSEC}} \times \frac{60\ \text{ASEC}}{\text{AMIN}} \times \frac{60\ \text{MIN}}{\text{DEG}} \times \frac{180°}{\pi\ \text{RAD}} = 1.031 \times 10^5\ \frac{\text{CYCLE}}{\text{RAD}} \quad (2)$$

AT THIS VALUE, $\mu_H(\lambda\Omega) = \frac{1}{e}$

TAKE NATURAL LOG OF (1)

$$\Rightarrow 1 = \frac{1}{2}\, 2.91\, k^2 \int_0^z C_N^2(z)\,dz\; (\lambda\Omega_{\frac{1}{e}})^{5/3}$$

LET $z \to \infty$ AND SOLVE FOR $\int_0^\infty C_N^2(z)\,dz$:

$$\int_0^\infty C_N^2(z)\,dz = \frac{2}{2.91\, k^2 (\lambda\Omega_{\frac{1}{e}})^{5/3}}$$

$$k = \frac{2\pi}{\lambda}$$

$$\Rightarrow \int_0^\infty C_N^2(z)\,dz = 2\left[2.91 \left(\frac{2\pi}{\lambda}\right)^2 (\lambda)^{5/3}\, \Omega_{\frac{1}{e}}^{5/3}\right]^{-1}$$

$$= 2\left[2.91\,(2\pi)^2\, \lambda^{-2}\, \lambda^{5/3}\, \Omega_{\frac{1}{e}}^{5/3}\right]^{-1}$$

$$= 2\left[2.91\,(2\pi)^2\, \lambda^{-\frac{1}{3}}\, \Omega_{\frac{1}{e}}^{5/3}\right]^{-1}$$

$$= \frac{2\,\lambda^{+\frac{1}{3}}}{2.91\,(2\pi)^2\, \Omega_{\frac{1}{e}}^{5/3}}$$

$$= 7.68 \times 10^{-11}\ \lambda^{\frac{1}{3}}$$

BUT WE ARE GIVEN $\lambda = 5 \times 10^{-7}\,m \quad (3)$

$$\Rightarrow \int_0^\infty C_N^2(z)\,dz = 6.09 \times 10^{-13}\ m^{\frac{1}{3}}$$

(CONT →)

b. GIVEN $C_N^2 = 10^{-15} m^{-2/3}$

USE THE PLUG ON 246

$$\Omega_{\frac{1}{e}} = \lambda^{4/5} / (57.44 \, C_N^2 \, z)^{3/5}$$

SOLVE FOR z:

$$z = \frac{1}{57.44 \, C_N^2} \; \frac{\lambda^{\frac{4}{3}}}{\Omega_e^{5/3}}$$

USE (2) & (3) TO GET

$z = 609$ m

Marks

EE 5308 - Takehome Midterm Results.

High = 93

Mean = 83.3

Low = 71

Grade distribution

93 (2)
89
83
78
76
71

Comments: A master solution is posted outside 260B. Please don't remove it - but feel free to look it over.

Note on Prob. #4: Part (a) Recall that the def. of A_c, the coherence area, is rather arbitrary. The correct approach here is to set $\mu(\Delta, 0) = 0$ and solve for ρ, since this really does imply incoherence of the light at ~~the~~ two pinholes.

Part (b) When $\mu(\Delta, 0)$ takes on its max. negative value, then we should ~~should~~ get the "most ~~destructive~~ interference" in the image plane. Now ~~the~~ μ is proportional to $\rho W J_0(\;)$ but the minimum occurs approx. where $J_0(\;)$ reaches its first max. negative value.

EE 5358 (JFW)

Name: **ROBERT J. MARKS II.**

(1)

Spring 1976
Open Book,
Take-Home Midterm Quiz

Due 12 noon, Saturday 3/20/76

Do all 4 problems. You must work independently! Show your assumptions and all your work. If questions of interpretation arise please see Dr. Walkup.
Please hand in these sheets with your exam!

22
15 ← (something seems to be missing here)
25
21

83

Prob. 1: Assume an optical source whose normalized power spectral density $\hat{\mathcal{G}}(\nu)$ has the negative exponential form. $(\Delta\nu \ll \bar{\nu})$

$$\hat{\mathcal{G}}(\nu) \cong \begin{cases} \dfrac{K_0}{\Delta\nu} \exp\left\{ -K_1 \left| \dfrac{\nu-\bar{\nu}}{\Delta\nu} \right| \right\} & , \nu > 0 \\ 0 & \text{———} \end{cases}$$

(a) What can you say about K_0 and K_1?

(b) What is $\gamma(\tau)$ for this source?

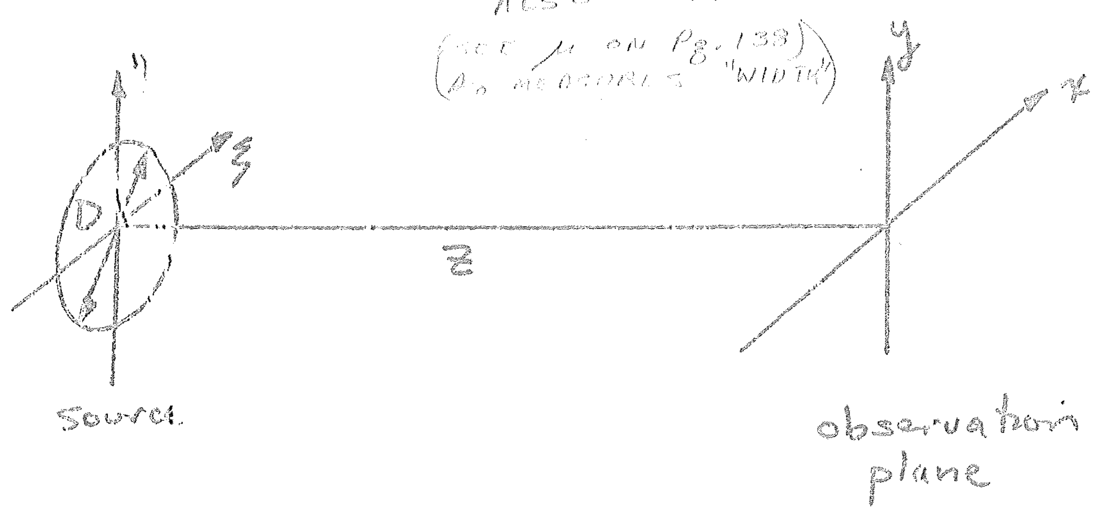
(c) If we evaluate τ_c, the coherence time for this source, what do we get? (i.e. evaluate τ_c).

(d) What fringe visibility will be observed [illegible]

Prob. 2: A certain partially coherent source, quasi monochromatic with mean wavelength $\bar{\lambda}$, can be described by the mutual intensity function

$$\underline{J}(\xi_1, \eta_1; \xi_2, \eta_2) \cong I_S(\xi_1, \eta_1)\,\underline{\mu}_{12}(\xi_1 - \xi_2, \eta_1 - \eta_2)$$

Assuming that $I_S(\xi_1, \eta_1)$ is constant over a circular disk of diameter D, and zero elsewhere, find an expression for the intensity distribution $I(x, y)$ in the observation plane shown below. ASSUME $\xi_2^2 - \xi_1^2 = (\xi_2 - \xi_1)(\xi_2 + \xi_1) \leq \Delta D \ll \lambda z$ (SHOW) (SAME FOR $\eta_2^2 - \eta_1^2$)
ALSO $\Delta \ll D$
(SEE μ ON Pg. 138)
(Δ MEASURES "WIDTH")



You may invoke the usual paraxial approximations. In addition, noting that $a^2 - b^2 = (a-b)(a+b)$, you should simplify your answer by assuming that

$$z \gg \frac{D\Delta}{\lambda}$$

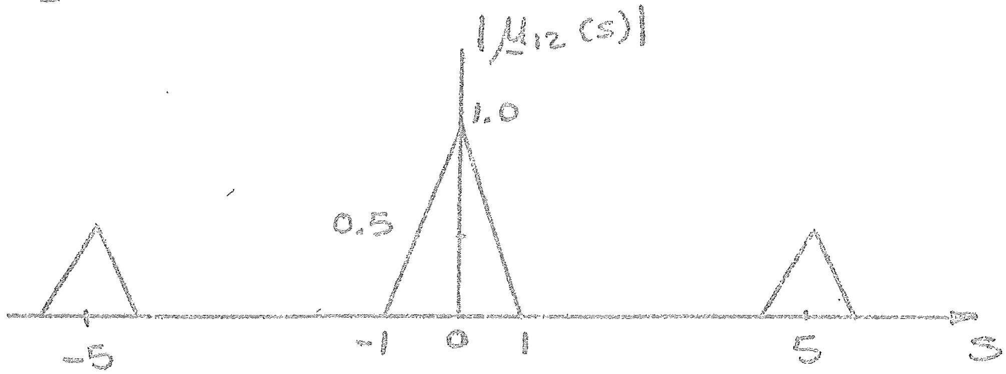
where Δ is the "width" of $\underline{\mu}_{12}(\Delta\xi, \Delta\eta)$.

FOR $\Delta \to 0$, ITS INCOHERENT
FOR $\Delta \to D$, COHERENT ACROSS [illegible],
DROP [illegible] IN EXPONENT

$\Delta \ll D$

(3)

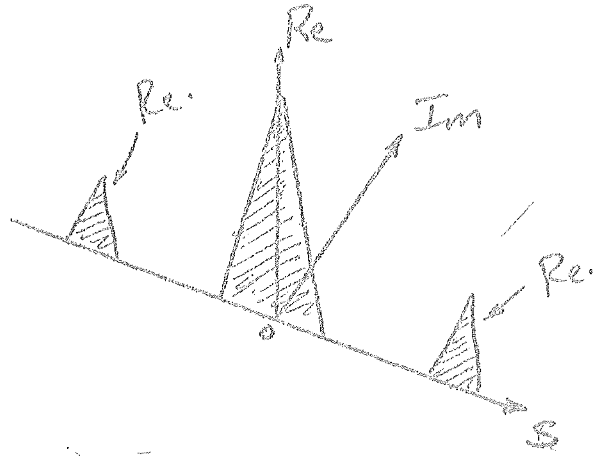
Prob. 3: In a certain one-dimensional imaging problem, a Michelson stellar interferometer measures the modulus $|\mu_{12}(s)|$ of the complex coherence factor over a range of spacings s. The result is as shown below:



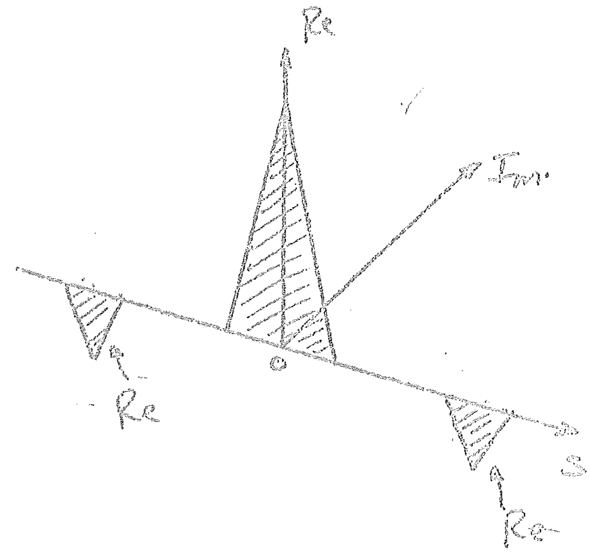
To know the intensity distribution of the incoherent source which gave rise to this curve, we must associate a phase distribution with $|\mu_{12}|$. Which of the following guesses at the full, complex-valued $\mu_{12}(s)$ could possibly be right, and which could not be? Why? Would *a priori* knowledge that the object is brightest at its center help you narrow down the choice?

Prob. #3 contd

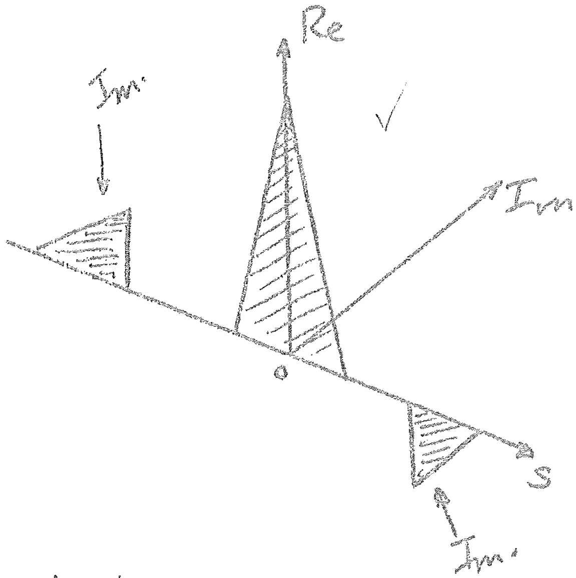
(a)



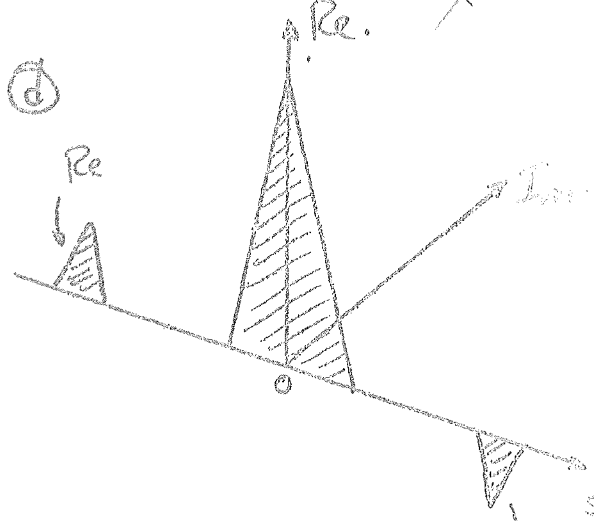
(b)



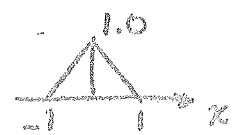
(c)



(d)



Helpful Hints:

$\mathcal{F}\{\Lambda(x)\} = \mathrm{sinc}^2 f$ where $\Lambda(x) =$ 

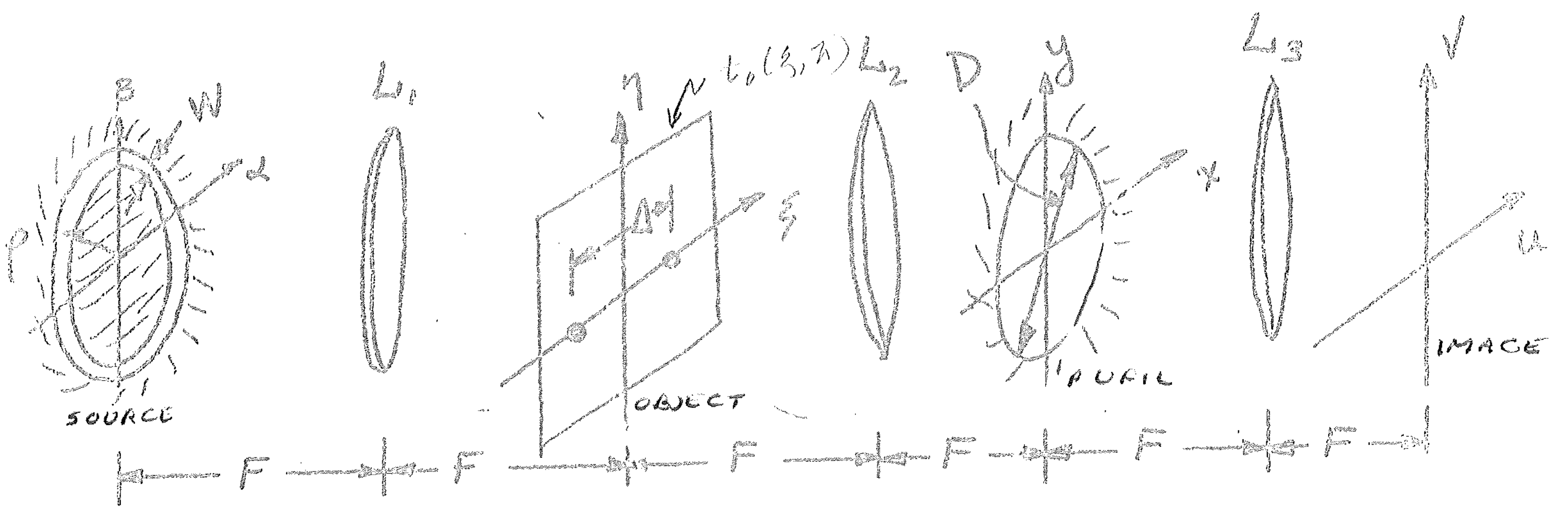
$\mathcal{F}\{\cos 2\pi a x\} = \frac{1}{2}\delta(f-a) + \frac{1}{2}\delta(f+a)$

$\mathcal{F}\{\sin 2\pi a x\} = \frac{j}{2}\delta(f-a) - \frac{j}{2}\delta(f+a)$

$\mathcal{F}\{e^{j 2\pi a x}\} = \delta(f+a)$

$\mathcal{F}\{g(x)\cdot h(x)\} = \mathcal{F}\{g\} * \mathcal{F}\{h\}$

Prob. #9 In the partially coherent imaging system shown below, the source is a thin incoherent annulus or ring, with mean radius ρ and radial width W.



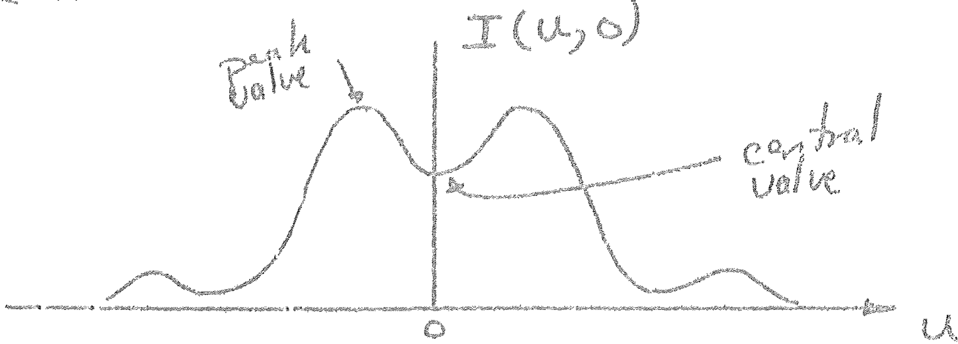
The pupil aperture is circular with diameter D, and the system is free of aberrations. The object consists of two tiny pinholes, individually unresolvable, lying on the ξ axis and separated by distance Δ. For this particular problem Δ is known to be given by $\Delta = \frac{1.22\lambda F}{D}$ where F is the common focal length of all the lenses. This distance Δ happens to be the so-called "Rayleigh distance" for which the peak of one "Airy pattern" (the diffraction-limited image of a pinhole) coincides exactly with the first zero of the second "Airy pattern."

Prob. 9 contd.

(a) Find the smallest radius ρ of the annular source for which the two pinholes are illuminated <u>incoherently</u>

(b) Find the radius ρ of the annular source for which the central value of the intensity in the partially resolved image drops to its smallest possible value relative to the peak value

(see illustration below).



Hint: The Fourier transform of a <u>thin</u> uniform annulus of mean radius ρ and width W is given approximately by

$$\underline{A}(\nu_x, \nu_y) \cong 2\pi\rho W J_0\left(2\pi\rho\sqrt{\nu_x^2+\nu_y^2}\right)$$

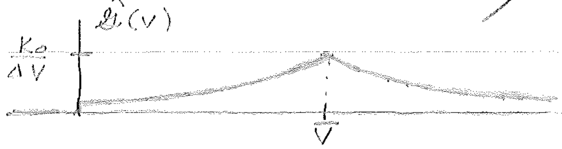
where J_0 is a Bessel function of the first kind, order zero.

ROBERT J. MARKS II.

(1-1)

1. $\hat{\mathcal{G}}(v) = \frac{K_0}{\Delta v} e^{-K_1 \left| \frac{v - \bar{v}}{\Delta v} \right|} \mu(v) \ni \mu(v) = \text{UNIT STEP}$

(ASSUME $\bar{v} > 0$, $\Delta v > 0$)



5

a. WE KNOW THAT

$$\int_0^\infty \hat{\mathcal{G}}(v)\,dv = 1$$

THUS

$$1 = \frac{K_0}{\Delta v} \int_0^{\bar{v}} e^{+K_1\left(\frac{v-\bar{v}}{\Delta v}\right)} dv + \frac{K_0}{\Delta v} \int_{\bar{v}}^\infty e^{-K_1\left(\frac{v-\bar{v}}{\Delta v}\right)} dv$$

$$= \frac{K_0}{\Delta v}\left(\frac{\Delta v}{K_1}\right) e^{K_1\left(\frac{v-\bar{v}}{\Delta v}\right)}\Big|_0^{\bar{v}} - \frac{K_0}{\Delta v}\left(\frac{\Delta v}{K_1}\right) e^{-K_1\left(\frac{v-\bar{v}}{\Delta v}\right)}\Big|_{\bar{v}}^\infty$$

$$= \frac{K_0}{K_1}\left[\left(1 - e^{-K_1\bar{v}/\Delta v}\right) - (0 - 1)\right]$$

$$= \frac{K_0}{K_1}\left[2 - e^{-K_1\bar{v}/\Delta v}\right]$$

$$\Rightarrow K_0 = \frac{K_1}{2 - e^{-K_1\bar{v}/\Delta v}} \qquad (1)$$

AND

$$\hat{\mathcal{G}}(v) = \frac{K_1}{\Delta v}\left(2 - e^{-K_1\bar{v}/\Delta v}\right)^{-1} e^{-K_1\left|\frac{v-\bar{v}}{\Delta v}\right|} \mu(v) \qquad (2)$$

ALSO

(1) SINCE $\hat{\mathcal{G}}(v)$ HAS UNITS OF $\frac{1}{\text{FREQUENCY}}$, K_0 IS UNITLESS

(2) K_1 IS UNITLESS

(3) IN ORDER THAT $\int_0^\infty \hat{\mathcal{G}}(v)\,dv$ CONVERGE, WE REQUIRE THAT $K_1 > 0$.

(4) SIMILARLY, $\int_0^\infty \hat{\mathcal{G}}(v)\,dv = 1 \Rightarrow K_0 > 0$ (THIS ALSO FOLLOWS FROM $K_0(2 - e^{-K_1\bar{v}/\Delta v}) = K_1$)

THUS, THE FORM OF Eq. 2 IS VALID FOR ALL $K_1 > 0$ (UNDER THE ASSUMPTIONS $\bar{v} > 0$ AND $\Delta v > 0$)

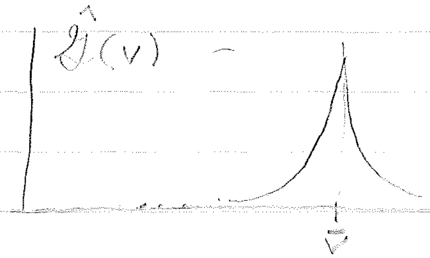
(CONT $\longrightarrow$)

WE MAY FURTHER SIMPLIFY $\hat{\mathcal{G}}(v)$ BY ASSUMING

a. $\bar{v} >> \Delta v \Rightarrow \frac{\bar{v}}{\Delta v} >> 1$

b. K_1 IS "REASONABLY" LARGE (CONSERVATIVELY $K_1 > 1$)
SO THAT $\frac{K_1 \bar{v}}{\Delta v} >> 1$

UNDER THESE TWO CONDITIONS, $\hat{\mathcal{G}}(v)$ TAKES ON MORE OF AN IMPULSIVE NATURE IN THE SENSE THAT $\hat{\mathcal{G}}(\bar{v}) >> \mathcal{G}(0)$ (MORE ON THIS LATER)



FURTHERMORE

$$K_0 = \frac{K_1}{2 - e^{-K_1 \bar{v}/\Delta v}} \simeq \frac{K_1}{2} \qquad (3)$$

AND

$$\hat{\mathcal{G}}(v) \simeq \frac{K_1}{2\Delta v} e^{-K_1 \left| \frac{v - \bar{v}}{\Delta v} \right|}$$

b. WE MAY EXPRESS THE COMPLEX DEGREE OF COHERENCE, $\underline{\gamma}(\tau)$, AS

$$\underline{\gamma}(\tau) = \int_0^{\infty} \hat{\mathcal{G}}(\nu)\, e^{-j 2\pi \nu \tau}\, d\nu$$

FOR OUR PROBLEM

$$\begin{aligned}
\underline{\gamma}(\tau) &= \frac{K_0}{\Delta\nu} \int_0^{\infty} e^{-K_1 \left| \frac{\nu - \bar{\nu}}{\Delta \nu} \right|} e^{-j 2\pi \nu \tau}\, d\nu \\
&= \frac{K_0}{\Delta\nu} \Big[\int_0^{\bar{\nu}} e^{K_1 \left(\frac{\nu - \bar{\nu}}{\Delta\nu} \right)} e^{-j 2\pi \nu \tau}\, d\nu \\
&\qquad + \int_{\bar{\nu}}^{\infty} e^{-K_1 \left(\frac{\nu - \bar{\nu}}{\Delta\nu} \right)} e^{-j 2\pi \nu \tau}\, d\nu \Big] \\
&= \frac{K_0}{\Delta\nu} \Big[e^{-\frac{K_1 \bar{\nu}}{\Delta\nu}} \int_0^{\bar{\nu}} e^{\left[\frac{K_1}{\Delta\nu} - j 2\pi\tau \right] \nu}\, d\nu \\
&\qquad + e^{\frac{K_1 \bar{\nu}}{\Delta\nu}} \int_{\bar{\nu}}^{\infty} e^{-\left[\frac{K_1}{\Delta\nu} + j 2\pi \tau \right] \nu}\, d\nu \Big] \\
&= \frac{K_0}{\Delta\nu} \Big[e^{-\frac{K_1 \bar{\nu}}{\Delta\nu}} \frac{1}{\frac{K_1}{\Delta\nu} - j 2\pi\tau} e^{\left[\frac{K_1}{\Delta\nu} - j 2\pi\tau \right] \nu} \Big|_0^{\bar{\nu}} \\
&\qquad - e^{\frac{K_1 \bar{\nu}}{\Delta\nu}} \frac{1}{\frac{K_1}{\Delta\nu} + j 2\pi\tau} e^{-\left[\frac{K_1}{\Delta\nu} + j 2\pi\tau \right] \nu} \Big|_{\bar{\nu}}^{\infty} \Big] \\
&= K_0 \Big[e^{-K_1 \bar{\nu}/\Delta\nu} \frac{1}{K_1 - j 2\pi \Delta\nu \tau} \left(e^{\left[\frac{K_1}{\Delta\nu} - j 2\pi\tau \right] \bar{\nu}} - 1 \right) \\
&\qquad - e^{K_1 \bar{\nu}/\Delta\nu} \frac{1}{K_1 + j 2\pi \Delta\nu \tau} \left(0 - e^{-\left[\frac{K_1}{\Delta\nu} + j 2\pi\tau \right] \bar{\nu}} \right) \Big] \\
&= K_0 \Big[\frac{1}{K_1 - j 2\pi \Delta\nu \tau} \left(e^{-j 2\pi \bar{\nu} \tau} - e^{-K_1 \bar{\nu}/\Delta\nu} \right) \qquad (4) \\
&\qquad + \frac{1}{K_1 + j 2\pi \Delta\nu \tau} e^{-j 2\pi \bar{\nu} \tau} \Big] \\
&= K_0 \left[\frac{1}{K_1 - j 2\pi \Delta\nu\tau} + \frac{1}{K_1 + j 2\pi \Delta\nu\tau} \right] e^{-j 2\pi \bar{\nu} \tau} \\
&\qquad - \frac{K_0}{K_1 - j 2\pi \Delta\nu \tau} e^{-K_1 \bar{\nu}/\Delta\nu} \\
&= \frac{2 K_0 K_1}{K_1^2 + (2\pi \Delta\nu \tau)^2} e^{-j 2\pi \bar{\nu} \tau} - \frac{K_0 (K_1 + j 2\pi \Delta\nu \tau)}{K_1^2 + (2\pi \Delta\nu\tau)^2} e^{-K_1 \bar{\nu}/\Delta\nu} \\
&= K_0 \left\{ K_1^2 + (2\pi \Delta\nu \tau)^2 \right\}^{-1} \left[2 K_1 e^{-j 2\pi \bar{\nu} \tau} - (K_1 + j 2\pi \Delta\nu \tau) e^{-K_1 \bar{\nu}/\Delta\nu} \right] \qquad (5)
\end{aligned}$$

THIS IS THE COMPLEX DEGREE OF COHERENCE ($\bar{\nu} > 0$, $\Delta\nu > 0$). SUBSTITUTING K_0 IN Eq. 1 :

$$\underline{\gamma}(\tau) = \frac{K_1}{\left(2 - e^{-K_1 \bar{\nu}/\Delta\nu}\right)} \left\{ K_1^2 + (2\pi\Delta\nu\tau)^2 \right\}^{-1} \left[2K_1 e^{-j 2\pi \bar{\nu} \tau} - (K_1 + j 2\pi \Delta\nu \tau) e^{-\frac{K_1 \bar{\nu}}{\Delta\nu}} \right] \qquad (5a)$$

(CONT →)

EMPLOYING THE ASSUMPTIONS IN PART 4 $\left(\frac{K_1\bar{V}}{\Delta V} >> 1\right)$, WE HAVE

$$e^{-K_1\bar{V}/\Delta V} \simeq 0 \qquad (6)$$

$$K_0 \simeq \frac{K_1}{2}$$

AND, FROM Eq. 5:

$$\underline{\gamma}(\tau) \simeq \frac{K_1}{2}\left\{K_1^2 + (2\pi\Delta V\tau)^2\right\}^{-1}\left(2K_1 e^{-j2\pi\bar{V}\tau}\right)$$

$$= \frac{K_1^2}{K_1^2+(2\pi\Delta V\tau)^2} e^{-j2\pi\bar{V}\tau} \qquad (7)$$

ON CLOSER INSPECTION OF Eq. 5, SOME QUESTION OF THE VALIDITY OF THE APPROXIMATION IN Eq. 6 ARISES (DUE TO THE MULTIPLICATIVE $j2\pi\tau$ TERM. THAT IS $\lim_{\tau\to\infty} j2\pi\tau\, e^{-K_1\bar{V}/\Delta V} = i\infty$ FOR ANY FINITE VALUE OF $\frac{K_1\bar{V}}{\Delta V}$). HOWEVER, THE TERM $[K_1^2+(2\pi\Delta V\tau)^2]^{-1}$ IN FRONT OF Eq. 5 BEATS THIS LIMIT TO ZERO. THE VALIDITY OF THE APPROXIMATION (Eq. 6) IS CLEARER IN Eq. 4.

EQUATION 7 IS REGOGNIZED AT THE FOURIER TRANSLORM OF $\hat{G}(V)$ WITHOUT THE UNIT STEP. THAT IS

$$\frac{K_1^2}{K_1^2+(2\pi\Delta V\tau)^2} e^{-j2\pi\bar{V}\tau} = \mathcal{F}^1\left[\frac{K_0}{\Delta V} e^{-K_1\left|\frac{V-\bar{V}}{\Delta V}\right|}\right]$$

$$= \int_{-\infty}^{\infty} \frac{K_0}{\Delta V} e^{-K_1\left|\frac{V-\bar{V}}{\Delta V}\right|} e^{-j2\pi V\tau} dV$$

WHERE $K_0 = \frac{K_1}{2}$ (CONT $\rightarrow$)

THUS, FOR THIS PARTICULAR PROBLEM, THE ASSUMPTION $\frac{K_1 \bar{V}}{\Delta V} \gg 1$ IS EQUIVALENT TO $\hat{\mathcal{D}}(\bar{V}) \gg \hat{\mathcal{D}}(0)$.

C. BY GOODMAN'S DEFN (Pg. 70):

5 $$\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau$$

LOOKING @ Eq. 5, THE RESULTING INTEGRAL FOR THIS RELATIONSHIP LOOKS RATHER UGLY. RECOGNIZING, THOUGH, THAT $\gamma(\tau) = \mathcal{F}^{-1}[\hat{\mathcal{G}}(\nu)]$, WE MAY CALL ON PARSEVAL'S THEOREM AND WRITE;

$$\begin{aligned}
\tau_c &= \int_{-\infty}^{\infty} |\hat{\mathcal{G}}(\nu)|^2 d\nu \\
&= \int_0^{\infty} |\hat{\mathcal{G}}(\nu)|^2 d\nu \\
&= \left(\frac{K_0}{\Delta\nu}\right)^2 \int_0^{\infty} e^{-2K_1\left|\frac{\nu-\bar{\nu}}{\Delta\nu}\right|} d\nu \\
&= \left(\frac{K_0}{\Delta\nu}\right)^2 \left[\int_0^{\bar{\nu}} e^{2K_1\left[\frac{\nu-\bar{\nu}}{\Delta\nu}\right]} d\nu + \int_{\bar{\nu}}^{\infty} e^{-2K_1\left[\frac{\nu-\bar{\nu}}{\Delta\nu}\right]} d\nu\right] \\
&= \left(\frac{K_0}{\Delta\nu}\right)^2 \left[\frac{\Delta\nu}{2K_1} e^{2K\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)}\Big|_0^{\bar{\nu}} - \frac{\Delta\nu}{2K_1} e^{-2K_1\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)}\Big|_{\bar{\nu}}^{\infty}\right] \\
&= \left(\frac{K_0}{\Delta\nu}\right)^2 \left(\frac{\Delta\nu}{2K_1}\right)\left[\left(1 - e^{-2K_1\bar{\nu}/\Delta\nu}\right) - (0-1)\right] \\
&= \frac{K_0^2}{2K_1\Delta\nu}\left(2 - e^{-2K_1\bar{\nu}/\Delta\nu}\right)
\end{aligned}$$

FIRST OFF, WE SHALL USE THE STRICT (NO ASSUMPTION) VALUE OF K_0 GIVEN IN Eq. 1:

$$\tau_c = \frac{K_1^2}{2K_1\Delta\nu}\,\frac{2 - e^{-2K_1\bar{\nu}/\Delta\nu}}{\left(2 - e^{-K_1\bar{\nu}/\Delta\nu}\right)^2} = \frac{K_1}{2\Delta\nu}\,\frac{2 - e^{-2K_1\bar{\nu}/\Delta\nu}}{\left(2 - e^{-K_1\bar{\nu}/\Delta\nu}\right)^2}$$

UNDER THE ASSUMPTION $\frac{K_1\bar{\nu}}{\Delta\nu} \gg 0$, WE HAVE $e^{-K_1\bar{\nu}/\Delta\nu} \simeq e^{-2K_1\bar{\nu}/\Delta\nu} \simeq 0$ AND

$$\tau_c \simeq \frac{K_1}{4\Delta\nu}$$

this is the one I'm looking for.

NOTE: (SEE Pg. 71) IF τ_c IS TO BE SAME ORDER OF MAGNITUDE OF $\Delta\nu^{-1}$, IT FOLLOWS THAT $K_1 \gg 0$ MUST BE "REASONABLY LARGE" AS PREVIOUSLY [illegible]

2

(1-6)

d. FOR THE MICHELSON INTERFEROMETER, THE VISIBILITY IS GIVEN AS (Pg. 21)

3 $\mathcal{V}(h) = \frac{2C_1C_2}{C_1^2+C_2^2}\ \gamma\left(\frac{2h}{c}\right)$ Now subst. in the results of (b)

WHERE C_1 AND C_2 ARE THE LOSSES IN THE RESPECTIVE ARMS OF THE INTERFEROMETER AND $\gamma(\tau) = |\underline{\gamma}(\tau)|$. NEAR $h=0$, THIS RELATIONSHIP BECOMES

near, not at

$\mathcal{V}(0) = C\ \gamma(0)$ I wanted an expression for $\mathcal{V}(h)$ near $h=0$ — $h=0$

WHERE $C = \frac{2C_1C_2}{C_1^2+C_2^2}$ (FOR EQUAL LOSSES, $C_1 = C_2$ AND $C = 1$).

THE "NO ASSUMPTION" VISIBILITY COMES FROM Eq. 5a (@ $\tau = 0$)

$\underline{\gamma}(0) = K_1\left[(2-e^{-K_1\bar{\nu}/\Delta\nu})K_1^2\right]^{-1}\left[2K_1 - K_1 e^{-\frac{K_1\bar{\nu}}{\Delta\nu}}\right]$

$= \frac{1}{(2-e^{-K_1\bar{\nu}/\Delta\nu})}\ (2-e^{-K_1\bar{\nu}/\Delta\nu})$

$= 1 = |\underline{\gamma}(0)| = \gamma(0)$

IT FOLLOWS THAT $\mathcal{V}(0) = C$. (SINCE A FURTHER APPROXIMATION WILL PROVIDE NO FURTHER SIMPLIFICATION, WE WON'T DO IT.) AGAIN, FOR THE CASE OF EQUAL ARM LOSSES, $\mathcal{V}(0) = 1$. RATHER A GOOD VISIBILITY! (True, but we knew that ~~the~~ $\gamma(0) = 1$ before starting!)

2. THE INTENSITY DISTRIBUTION ON THE X-Y PLANE CAN BE WRITTEN AS (Pg. 111)

15

$$I(x,y) = \iint_{\xi_1 \eta_1} \iint_{\xi_2 \eta_2} J(\xi_1, \eta_1; \xi_2, \eta_2)\, e^{-j\frac{2\pi}{\lambda}(r_2 - r_1)} \frac{\chi(\theta_1)}{\lambda r_1} \frac{\chi(\theta_2)}{\lambda r_2}\, ds_1 ds_2$$

WHERE $ds_i = d\eta_i d\xi_i$.

SINCE $J(\xi_1, \eta_1; \xi_2, \eta_2) = I_S(\xi_1, \eta_1)\, \mu_{12}(\Delta\xi, \Delta\eta)$

WHERE $\Delta\xi = \xi_1 - \xi_2$ AND $\Delta\eta = \eta_1 - \eta_2$;

$$I(x,y) = \underset{(\Sigma_1)}{\iint_{\xi_1 \eta_1}} \underset{(\Sigma_2)}{\iint_{\xi_2 \eta_2}} I_S(\xi_1, \eta_1)\, \mu_{12}(\Delta\xi, \Delta\eta)\, e^{-j\frac{2\pi}{\lambda}(r_2 - r_1)} \frac{\chi(\theta_1)}{\lambda r_1} \frac{\chi(\theta_2)}{\lambda r_2}\, ds_1 ds_2$$

MAKE THE USUAL APPROXIMATION

$$\chi(\theta_1) \simeq \chi(\theta_2) \simeq 1$$

$$r_1 \simeq r_2 \simeq z$$

$$\Rightarrow I(x,y) = \frac{1}{(\lambda z)^2} \iint_{\xi_1 \eta_1} \iint_{\xi_2 \eta_2} I(\xi_1, \eta_1)\, \mu_{12}(\Delta\xi, \Delta\eta)\, e^{-j\frac{2\pi}{\lambda}(r_2 - r_1)}\, ds_1 ds_2$$

AND THE PARAXIAL APPROXIMATION: (Pg. 89)

$$r_2 - r_1 \cong \frac{1}{2z}\left[\rho_2^2 - \rho_1^2 + 2\Delta\xi X + 2\Delta\eta Y\right]$$

$$\Rightarrow I(x,y) = \frac{1}{(\lambda z)^2} \iiiint I(\xi_1, \eta_1)\, \mu_{12}(\Delta\xi, \Delta\eta)\, e^{-j\frac{\pi}{\lambda z}(\rho_2^2 - \rho_1^2 + 2\Delta\xi X + 2\Delta\eta Y)}\, ds_1 ds_2$$

WHERE

$$\rho_2^2 = \eta_2^2 + \xi_2^2$$

$$\rho_1^2 = \eta_1^2 + \xi_1^2$$

(CONT →)

WE ARE GIVEN THAT $I(\xi_1, \eta_1)$ IS UNIFORM OVER Σ_1. LET

$$I(x,y) = \frac{I_0}{(\lambda z)^2} \iint_{\Sigma_1} \iint_{\Sigma_2} \mu_{12}(\Delta\xi, \Delta\eta)\, e^{-j\frac{\pi}{\lambda z}(\rho_2^2 - \rho_1^2 + 2\Delta\xi x + 2\Delta\eta y)}\, ds_1\, ds_2$$

FURTHERMORE, BOTH Σ_1 & Σ_2 CORRESPOND TO A CIRCLE FUNCTION $\mathrm{circ}\ ^{2\rho_i}/_{D_0}$. WE EXPLICITLY STATE THIS, AND LET THE INTEGRAL LIMITS GO FROM $-\infty$ TO ∞:

$$I(x,y) = \frac{I_0}{(\lambda z)^2} \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \mathrm{circ}\left(\frac{2\rho_1}{D}\right) \mathrm{circ}\left(\frac{2\rho_2}{D}\right) \quad (1)$$
$$\times \mu_{12}(\Delta\xi, \Delta\eta)\, e^{-j\frac{\pi}{\lambda z}(\rho_2^2 - \rho_1^2 + 2\Delta\xi x + 2\Delta\eta y)}\, ds_1\, ds_2$$

NOW

$$z \gg {}^{D\Delta}/_{\lambda} \quad (2)$$

CONSIDER, THEN, THE EXPONENT TERM IN Eq. 1:

$$\begin{aligned}\rho_2^2 - \rho_1^2 &= (\xi_2^2 + \eta_2^2) - (\xi_1^2 + \xi_2^2)\\ &= (\xi_2^2 - \xi_1^2) + (\eta_2^2 - \eta_1^2)\\ &= (\xi_2 - \xi_1)(\xi_2 + \xi_1) + (\eta_2 - \eta_1)(\eta_2 + \eta_1)\\ &= -\Delta\xi\,(\xi_2 + \xi_1) - \Delta\eta\,(\eta_2 + \eta_1)\end{aligned}$$

quite a bit more simplific is possible here. −10

Is there more?

?

(CONT →)

25

3. WE KNOW FROM THE VAN CITTERT - ZERNIKE THEOREM THAT THE FOURIER TRANSFORM OF THE INTENSITY IS PROPORTIONAL TO THE COMPLEX COHERENCE FACTOR (FOR AN INCOHERENT SOURCE) TO WITHIN A PHASE FACTOR $e^{j\psi}$ (Pg. 131).
SINCE WE ARE USING A MICHELSON STELLAR INTERFEROMETER (ie WE ARE LOOKING AT STARS), $z \gg 2(\rho_2^2 - \rho_1^2)/\lambda$, AND, BY CONDITION (1) ON Pg. 135, WE MAY SET $e^{-j\psi} \approx 1$, LEAVING A DIRECT PROPORTIONALITY (IN ONE DIMENSION) OF:

$$\mu(\Delta x) = \frac{1}{E} \int_{-\infty}^{\infty} I(\xi)\, e^{j \frac{2\pi}{\lambda z} \xi \Delta x}\, d\xi$$

WHERE $\Delta x = x_1 - x_2 = s =$ "PINHOLE" SPACING, AND

$$E = \int_{-\infty}^{\infty} I(\xi)\, d\xi > 0$$

(THIS RELATIONSHIP IS GIVEN ON BOTTOM OF 131, $\psi = 0$)
SETTING $v = s/\lambda z$, WE HAVE

$$\mu(v) = \frac{1}{E} \int_{-\infty}^{\infty} I(\xi)\, e^{j 2\pi v \xi}\, d\xi$$
$$= \frac{1}{E} \mathcal{F}^{+1}\left[I(\xi) \right]$$

WHERE $\mathcal{F}^{+1}(\cdot)$ DENOTES FOURIER TRANSFORMATION.
IT FOLLOWS THAT

$$I(\xi) = E\, \mathcal{F}^{-1}\left[\mu(v)\right] \triangleq E \int_{-\infty}^{\infty} \mu(v)\, e^{-j 2\pi v \xi}\, dv$$

(1)

(CONT →)

NOW TO THE PROBLEM. WE KNOW THAT $I(\xi)$ IS NON-NEGATIVE AND REAL. THUS $\mu(\nu)$ MUST BE HERMETIAN:

$$\mu(\nu) = \mu^*(-\nu)$$

THIS IMMEDIATELY RULES OUT $\mu_d(s)$ AS PICTURED IN FIG. d OF THE PROBLEM STATEMENT. (ie, IT ISN'T HERMETIAN) ✓

WE MUST NOW CHECK a, b, AND c TO SEE IF THEY ARE NON-NEGATIVE DEFINATE. * (FOOTNOTE, NEXT PGE)

a. $\mu_a(s) = \Lambda(s) + \frac{1}{2}\left[\Lambda(s+5) + \Lambda(s-5)\right]$

$$\Rightarrow \mu_a(\nu) = \Lambda(\nu) + \frac{1}{2}\left[\Lambda(\nu+5) + \Lambda(\nu-5)\right]$$

THUS, FROM Eq. 1:

$$\frac{1}{E} I_a(\xi) = \operatorname{sinc}^2(\xi) + \frac{1}{2}\left[\operatorname{sinc}^2\xi \, e^{+j2\pi(5)\xi} + \operatorname{sinc}^2\xi \, e^{-j2\pi(5)\xi}\right]$$

$$= \operatorname{sinc}^2(\xi)\left[1 + \cos 10\pi\xi\right] \checkmark$$

$$= 2\operatorname{sinc}^2(\xi)\cos^2 5\pi\xi$$

$$\Rightarrow I_a(\xi) = 2E\operatorname{sinc}^2(\xi)\cos^2(5\pi\xi) \geq 0 \qquad (2)$$

THUS, THE $\mu_a(s)$ IN FIG. a COULD VERY WELL BE THE ACTUAL COMPLEX COHERENCE FACTOR, AND $I_a(\xi)$ THE CORRESPONDING INTENSITY.

(CONT. →)

b. $\mu_b(\nu) = \Lambda(\nu) - \frac{1}{2}[\Lambda(\nu+5) + \Lambda(\nu-5)]$

GOING THROUGH THE STEPS IN PART a:

$\frac{1}{E} I_b(\xi) = \text{sinc}^2(\xi)\,[1 - \cos 10\pi\xi]$

$\Rightarrow I_b(\xi) = 2E\,\text{sinc}^2(\xi)\,\sin^2(5\pi\xi) \geq 0$ (3)

THUS, $\mu_b(s)$ COULD ALSO BE THE ACTUAL COMPLEX COHERENCE FACTOR

c. $\mu_c(\nu) = \Lambda(\nu) + \frac{i}{2}[\Lambda(\nu+5) - \Lambda(\nu-5)]$

$\frac{1}{E} I_c(\xi) = \text{sinc}^2(\xi) + \frac{i}{2}[\text{sinc}^2(\xi)e^{j2\pi(5)\xi} - \text{sinc}^2(\xi)e^{-j2\pi(5)\xi}]$

$= \text{sinc}^2\xi - \text{sinc}^2\xi\,\frac{1}{i2}[e^{j10\pi\xi} - e^{-j10\pi\xi}]$

$= \text{sinc}^2\xi - \text{sinc}^2\xi\,\sin 10\pi\xi$

$\Rightarrow I_c(\xi) = E\,\text{sinc}^2\xi\,[1 - \sin 10\pi\xi] \geq 0$ (4)

(SINCE $0 \leq 1 - \sin 10\pi\xi \leq 2$)

AGAIN, $\mu_c(s)$ COULD BE THE ACTUAL COMPLEX COHERENCE FACTOR.

ACTUALLY, THE WORK UP TO THIS POINT COULD HAVE BEEN AVOIDED, SINCE ALL HERMETIAN FUNCTIONS ARE NON-NEGATIVE DEFINITE*, AND, BY INSPECTION, a, b, AND c ARE HERMETIAN.

*[A NECESSARY CONDITION FOR $\mu(\nu)$ TO BE NON-NEGATIVE DEFINATE IS THAT $\mathcal{F}^{-1}[\mu(\nu)] \geq 0$ FOR ALL ξ]

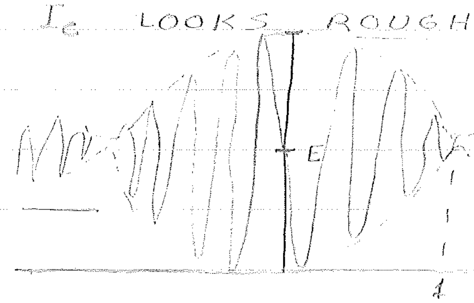
(CONT →)

WE MAY, HOWEVER, USE THESE RESULTS TO ANSWER THE SECOND PART OF THE PROBLEM. FROM Eq.S. (2), (3), AND (4), WE WRITE

$$I_a(0) = 2E$$
$$I_b(0) = 0$$
$$I_c(0) = E$$

THUS, OF THESE THREE REMAINING CHOICES, I_a IS BRIGHTEST IN IT'S CENTER ($\xi = 0$). THUS, IF WE WERE TO CHOOSE BETWEEN μ_a, μ_b, AND μ_c, WITH THE APRIORI KNOWLEDGE THAT THE OBJECT IS THE BRIGHTEST IN ITS CENTER, WE WOULD HAVE TO CHOOSE μ_a.

NOTE: I_c LOOKS ROUGHLY LIKE



AND OBVIOUSLY IS NOT BRIGHTEST IS ITS CENTER (ie $I_c(0) < I_{c(MAX)}$)

21

4.a. WE HAVE SHOWN IN PROBLEM #4 OF HOMEWORK SET #4, THAT WHEN AN INCOHERENT SOURCE WITH INTENSITY $I(\alpha,\beta)$ IS PLACED IN THE FRONT FOCAL PLANE OF A THIN LENS, THEN THE COMPLEX COHERENCE FACTOR IN THE REAR FOCAL PLANE IS GIVEN BY

$$\mu(\Delta\xi,\Delta\eta) = \frac{1}{E}\int\!\!\int_{-\infty}^{\infty} I(\alpha,\beta)\, e^{j\frac{2\pi}{\lambda F}(\Delta\xi\alpha+\Delta\eta\beta)}\, d\alpha\, d\beta \quad ①$$

WHERE

$$E = \int\!\!\int_{-\infty}^{\infty} I(\alpha,\beta)\, d\alpha\, d\beta \qquad ②^{*}$$

$$\Delta\xi = \xi_1 - \xi_2 \qquad \Delta\eta = \eta_1 - \eta_2$$

AND (ξ,η) IS THE REAR FOCAL PLANE (FOR THIS PROBLEM, THE OBJECT PLANE)

WE MAY EQUIVALENTLY WRITE Eq. 1 AS

$$\mu(\nu_x,\nu_y) = \frac{1}{E}\int\!\!\int_{-\infty}^{\infty} I(\alpha,\beta)\, e^{j2\pi(\alpha\nu_x+\beta\nu_y)}\, d\alpha\, d\beta$$

$$= \frac{1}{E}\mathcal{F}\left[I(\alpha,\beta)\right] \quad ; \; \nu_x = \frac{\Delta\xi}{\lambda f}, \; \nu_y = \frac{\Delta\eta}{\lambda f}$$

FOR THE GIVEN UNIFORM THIN ANNULUS, WE HAVE BEEN INFORMED THAT

$$\mathcal{F}\left[I(\alpha,\beta)\right] \cong 2\pi\rho\, W\, J_0\left(2\pi\rho\sqrt{\nu_x^2+\nu_y^2}\right)$$

(CONT →)

* FOR E TO BE CONSTANT, IT IS NECESSARY TO REQUIRE W BE INVERSELY PROPORTIONAL TO ρ (SINCE $\int\int I(\alpha,\beta)\,d\alpha\,d\beta \cong I_0 W(2\pi\rho)$)

ASSUMING THAT THIS APPROXIMATION IS PRETTY GOOD, WE WRITE

$$\mu(V_x, V_y) \approx \frac{2\pi\rho W}{E} J_0\left[2\pi\rho\sqrt{V_x^2 + V_y^2}\right]$$

OR

$$\mu(\Delta\xi, \Delta\eta) \approx \frac{2\pi\rho W}{E} J_0\left(\frac{2\pi\rho}{\lambda F}\sqrt{\xi^2+\eta^2}\right)$$

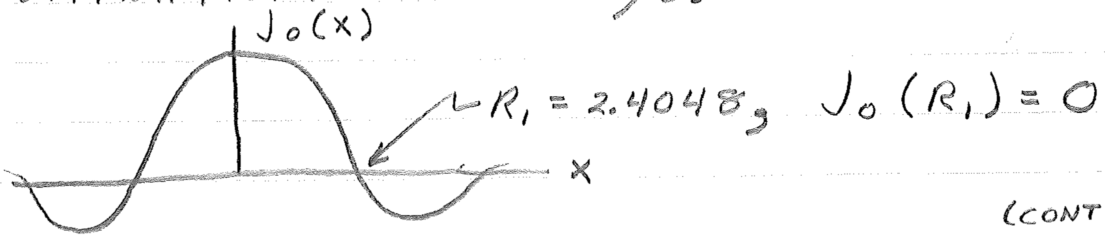
$$= \frac{2\pi\rho W}{E} J_0\left(\frac{2\pi\rho r}{\lambda F}\right) \qquad (3)$$

WHERE $r = \sqrt{\xi^2 + \eta^2}$

WITH REFERENCE TO THE GEOMETRY, WE WISH TO FIND A ρ SUCH THAT THE LIGHT FALLING ON THE PINHOLES ON THE ξ AXIS IS INCOHERENT. WE SAY THE LIGHT IS INCOHERENT WHEN $\mu = \mathcal{V}$ = VISIBILITY = 0. FROM Eq. 3, THIS OCCURS WHEN

$$J_0\left(\frac{2\pi\rho r}{\lambda F}\right) = 0$$

FROM THE LOWLY CRC, WE FIND THAT $R_1 = 2.4048$ GIVES THE FIRST ZERO OF $J_0(x)$. THIS WILL GIVE THE SMALLEST VALUE OF ρ FOR INCOHERENT ILLUMINATION. CALL IT ρ_0.



(CONT →)

THUS, WE HAVE

$$\rho_0 = \frac{\lambda F R_1}{2\pi r}$$

BUT! SINCE $\Delta\eta = 0$, WE HAVE

$r = \Delta\xi \cdot$ $\Delta\xi = \Delta$, not $\frac{\Delta}{2}$

$x = \frac{1}{2}\Delta = \frac{1.22\lambda F}{2D}$

THUS

$$\rho_0 = \left(\frac{\lambda F R_1}{2\pi}\right)\left(\frac{2D}{1.22\lambda F}\right)$$

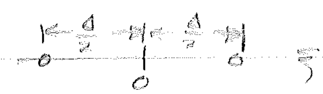
-2

$$= \frac{DR_1}{1.22\pi} = \frac{D(2.40)}{1.22\pi}$$

$$= 0.626\ D$$

correct answer $\approx 0.32\ D$ (half your result)

(DUE TO SYMMETRY WE RESTRICT Δ TO BE EVENLY SPACED ABOUT THE ORIGIN:



)

b. THIS PROBLEM IS ADDRESSED IN SEC. 3.3.3. WE NEED MERELY TO COPY DOWN SOME RELATIONSHIPS AND INTERPRET.

THE OBJECT TRANSPARANCY IS

$$t_o(\xi,\eta) = a\,\delta(\xi-\tfrac{\Delta}{2};\eta) + a\,\delta(\xi+\tfrac{\Delta}{2},\eta)$$

THE CORRESPONDING IMAGE PLANE INTENSITY (Pg. 163) IS

$$I_i(u,v) = I\Big[\,|\underline{K}(u-\tfrac{\Delta}{2},v)|^2 + |\underline{K}(u+\tfrac{\Delta}{2};v)|^2 + 2\,Re\{\underline{\mu}\,\underline{K}(u-\tfrac{\Delta}{2};v)\,\underline{K}^*(u+\tfrac{\Delta}{2};v)\}$$

WHERE $I = a^2\,\underline{J}_o(0,0)$ ("o" FOR OBJECT)

$$\underline{\mu} = \frac{1}{I}\,a^2\,\underline{J}_o(\Delta,0)$$

FOR A CIRCULAR PUPIL FUNCTION (ABBERRATION-FREE), WE HAVE (FROM 164)

$$\underline{K}(u,v) = K(s) \overset{\text{REAL}}{=} \frac{\bar{k}D^2}{8F}\left[2\,\frac{J_1\left(\frac{\bar{k}Ds}{2F}\right)}{\frac{\bar{k}D}{2F}s}\right]$$

WHERE $s = \sqrt{u^2+v^2}$ AND D IS THE PUPIL'S DIAMETER. IT FOLLOWS THAT

$$\underline{K}(u\pm\tfrac{\Delta}{2};0) = K(u\pm\tfrac{\Delta}{2})$$

$$= \frac{\bar{k}D^2}{8F}\left[2\,\frac{J_1\left\{\frac{\bar{k}D(u\pm\frac{\Delta}{2})}{2F}\right\}}{\frac{\bar{k}D\,(u\pm\frac{\Delta}{2})}{2F}}\right] \qquad (4)$$

(CONT →)

WE MAY REWRITE THE INTENSITY (@ $V=0$)
(VIA THE Eq. @ THE TOP OF 165)

$$I_i(U,0) = I\left[K^2(U-\tfrac{\Delta}{2}) + K^2(U+\tfrac{\Delta}{2}) + 2\mu K(U-\tfrac{\Delta}{2})\, K(U+\tfrac{\Delta}{2})\cos\phi \right] \quad (5)$$

WHERE

$$\mu = |\mu| \quad \text{AND} \quad \phi = \arg \mu$$

THIS RELATIONSHIP FOLLOWS, SINCE K IS STRICTLY REAL.

WE NOW WISH TO FIND A ρ_1, SUCH THAT $I_{i\,MAX}/I_i(0,0)$ IS MAXIMUM. FIRST OFF, WE NOTE THAT I WILL NOT ENTER INTO THE CALCULATION. SECONDLY, $K^2(U \pm \frac{\Delta}{2})$ IS INDEPENDENT OF OUR SOURCE PARAMETERS. THUS, WE CONCLUDE, THAT THE PARAMETER OF INTEREST IN Eq. 5 IS μ, HERE REWRITTEN FROM Eq. 3:

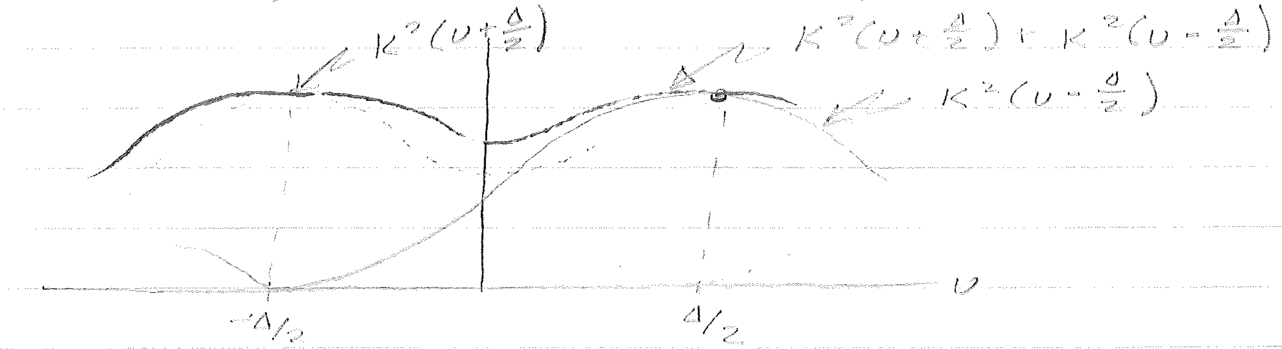
$$\mu = \frac{2\pi\rho W}{E} J_0\left(\frac{2\pi\rho r}{\lambda F}\right)$$

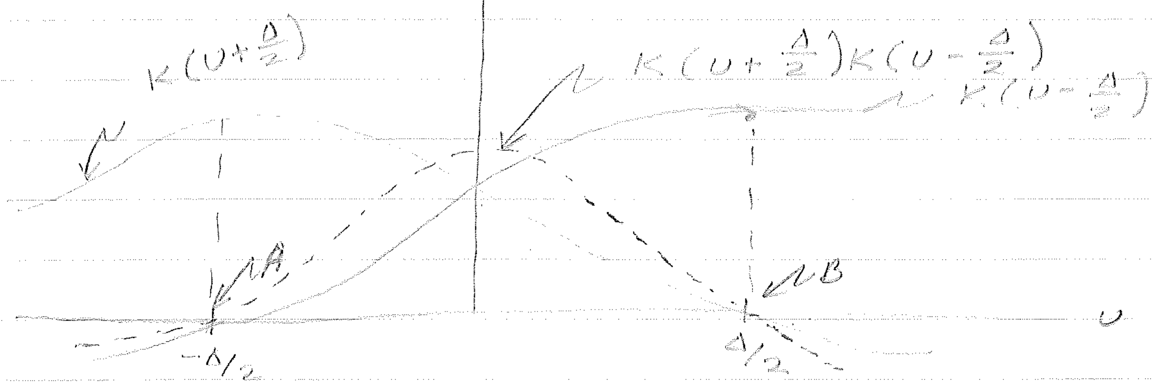
IN Eq. 5, WE ARE INTERESTED IN μ EVALUATED @ $r = \frac{\Delta}{2}$

$$\Rightarrow \mu = \frac{2\pi\rho W}{E} J_0\left(\frac{\pi\rho\Delta}{\lambda F}\right) = \frac{2\pi\rho W}{E} J_0\left(\frac{1.22\,\pi\rho}{D}\right) \quad (6)$$

(CONT →)

ROUGHLY, A SCETCH OF $I(U,O)$'S TERMS ARE





FOR FIXED I, $I_i(\pm\frac{\Delta}{2})$ REMAINS FIXED. BY VARYING THE AMPLITUDE OF THE BOTTOM SKETCH (BY $Re\,\mu$), WE GENERATE AN INTENSITY CURVE MUCH LIKE THAT ON Pg. 165. (THE TOP SKETCH ABOVE REMAINS FIXED).

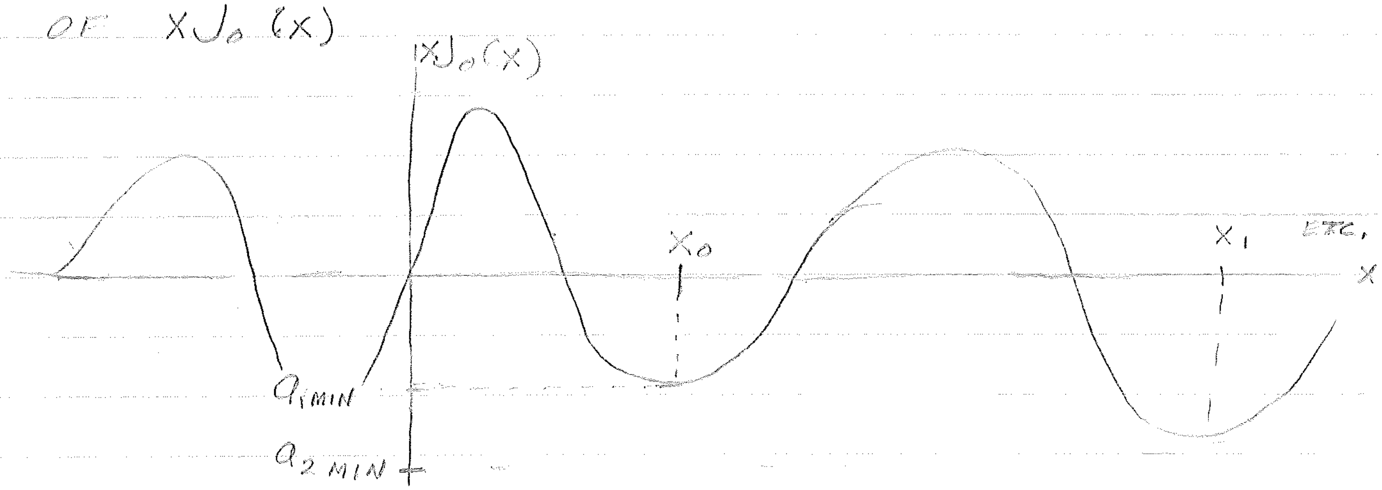
IT WOULD BE NICE TO SAY THAT THE RELATIVE MAXIMA OF THIS INTENSITY CURVE (FOR SUFFICIENTLY SMALL $Re\,\mu$) LIES @ $U = \pm\Delta/2$. THE ACTUAL MAXIMA, HOWEVER, WILL BE A WEE BIT MORE CLOSELY SPACED. (FOR CURVES AS SHOWN) TO DETERMINE THESE VALUES EXACTLY WOULD REQUIRE DIFFERINTIATING

(CONT →)

SOMBRARO FUNCTIONS. (A BLEAK UNDERTAKING). AS SUCH, AS ENGINEERS, LETS ASSUME THAT THE SLOPES AT POINTS A AND B ARE SUFFICIENTLY SMALL, AND THAT THE RELATIVE MAXIMA (IF ANY), OCCUR @ $U = \pm \frac{\Delta}{2}$. (THERE WOULD ALSO BE A MAXIMA @ $U=0$ FOR LARGE ENOUGH $Re\,\mu$).

IT REMAINS TO MINIMIZE THE TERM $\mu\, K(U+\frac{\Delta}{2})\, K(U-\frac{\Delta}{2})$ @ $U=0$ (HERE μ IS REAL) ONE'S FIRST INCLINATION IS TO SET $\mu = 0$ (COMPLETE INCOHERENCE), BUT WE CAN DO BETTER THAN THAT (i.e. SET $\mu < 0$). WITH REFERENCE TO Eq. 6, THE SMALLEST VALUES OF μ COME FROM THE NEGATIVE EXTREMA OF $XJ_0(X)$

$|\mu| = \mu$
($\cos\phi = -1$)



FROM CRC

$X_0 = 4.1$ AND $a_{MIN} = 1.6$

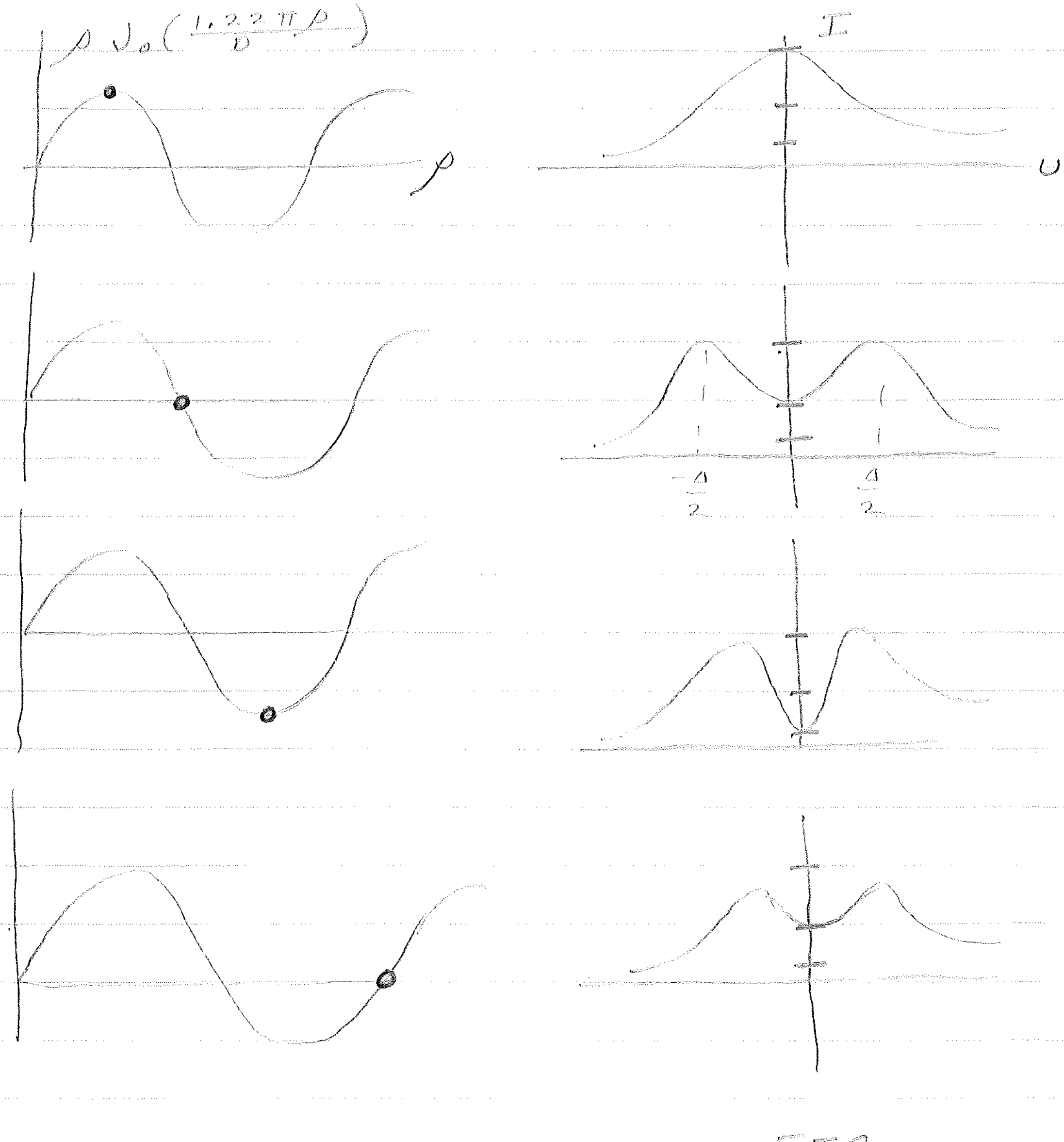
BUT $a_{MIN_2} < a_{MIN_1}$

$\Rightarrow$ MAGNITUDE OF MINIMUMS INCREASE WITH X

(CONT →)

SO WHATS THE ANSWER TO THE PROBLEM?
THERE IS NONE.

ROUGHLY, ρ MAY BE THOUGHT OF AS X ON THE PREVIOUS SCHETCH, AS ρ (>0) GOES FROM PEAK TO PEAK, THE INTENSITY (@ $U=0$) GOES THOUGH AN OSCILLATION OF SORTS:



ETC.

(CONT →)

FURTHERMORE, THE VALUE OF $I_i(0)$ BECOMES LESS AND LESS FOR EACH VALUE OF INCREASING ρ CORRESPONDING TO A RELATIVE MINIMUM. FOR LARGE ρ, $\rho J_0(\rho)$ ACTS PURELY SINUSOIDAL.

TO ACTUALLY SOLVE THIS PROBLEM, ONE WOULD NEED TO KNOW SOME CONSTRAINTS ON THE SYSTEM (SUCH AS PUPIL SIZES) TO UPPER BOUND ALLOWABLE ρ'S. THE GIVEN PROBLEM STATEMENT IS MEANINGLESS (UNLESS I MISS SOMETHING).

SUPPOSE WE WISH TO CALCULATE THE FIRST MINIMUM, CORRESPONDING TO $X_0 = 4.1$. THEN, FROM Eq. 6:

$$X_0 = 4.1 = \frac{1.22\pi\rho}{D}$$

−2

THE DESIRED ρ IS

$$\rho = \frac{D}{\pi}\,\frac{4.1}{1.22} \approx \frac{3.4}{\pi} D \approx 1.1 D$$

$\rho \approx 0.5D$

So again you're too high by a factor of two.

Bob - I think you're making part (b) appear much tougher than it is. It seems to me that all one needs to ask is for what ρ ~~is~~ does $M_{12}(\Delta, 0)$ have its max negative value. $J_0(x) \approx -0.4$ for $x \approx 3.8$, then plug in.

EE 6358 (JFW)
Spring 1976

Project

1. Read 2-3 basic papers in an area of your choice (choose from journals, monographs, etc.). See sample topics below.

2. Write an 8-10 page paper summarizing the key results and indicating the state-of-the-art in the area. Paper due on April 23, 1976

3. Present a 20 minute illustrated talk to the class based on your research efforts. The talks will be presented during the week of April 26, 1976

Sample topics:

1. speckle interferometry
2. info. processing with speckle
3. active optics for imaging through the atmosphere
4. suppression of optical noise sources
5. synthetic aperture imaging

To: Interested Faculty/Students

From: John Walkup

Subject: EE 5353 (Statistical Optics) Oral Talks

The members of the class are going to be giving 20 minute oral talks on various areas of statistical optics. These talks will be presented in the EE Conference Room (Bullen Room- EE 105) on MWF, 9:30 am of the week of (26-30 April. The talks will be illustrated and should (we hope) be informative. Please feel free to sit in if interested.

Monday 4/26:	Steven Bell -	"Introduction to Superresolution"
9:30-10:30 am	Bob Marks -	"Degrees of Freedom of an Image: Superresolution and Effects of Coherence"

Wednesday 4/28:

9:30-11:00 am	Ajit Kwatra-	"Interferometric Star-Tracking"
	Doug Brandon-	"Stellar Speckle Interferometry"
	Lloyd Matthews-	"Propagation of Intensity through the Turbulent Atmosphere"

Friday 4/30:

9:30-10:30 am	Hong Yao-	"Measurements of Atmospheric Turbulence"
	Ching-Tsai Pan-	"Measurement of Surface Roughness using Speckle"

The Degrees of Freedom of an Image: Superresolution and Effects of Coherence

Robert J. Marks II

The number of degrees of freedom of an image is a rough measure of the amount of information contained in the image. From a communications viewpoint, if one were to transmit a signal with a certain number of degrees of freedom (DOF), the optimal receiver would receive the same amount of information. Noise and channel capacity, though, many times limit such a performance.

In optics, the signal to be sent is termed the object. The channel consists of the system geometry (including aberrations) and appropriate channel noise. The receiver is simply an output plane whereon the image is received. The capacity of an imaging system rests on the channel and upon the mode of transmittance, or, more specifically, the degree of coherence of the (quasi-monochromatic) light.

This paper traces the development of the concept of degrees of freedom as applied to imaging systems, and particularly addresses the role of coherence on determining the system's information capacity. For effects of aberrations, the motivated reader is referred to Gori et. al.[1] and for effects of noise, to Bendinelli et. al.[2]

The concept of degrees of freedom

The idea of DOF, as applied to optics, was probably first suggested by di Francia[3] in 1955. He considered the simple one dimensional imaging system pictured in Fig. 1. An object is illuminated by a normally incident coherent plane wave and is Fourier transformed by L1. On the back focal plane of L1, there is a pupil of spatial width 2S. The spectrum of the

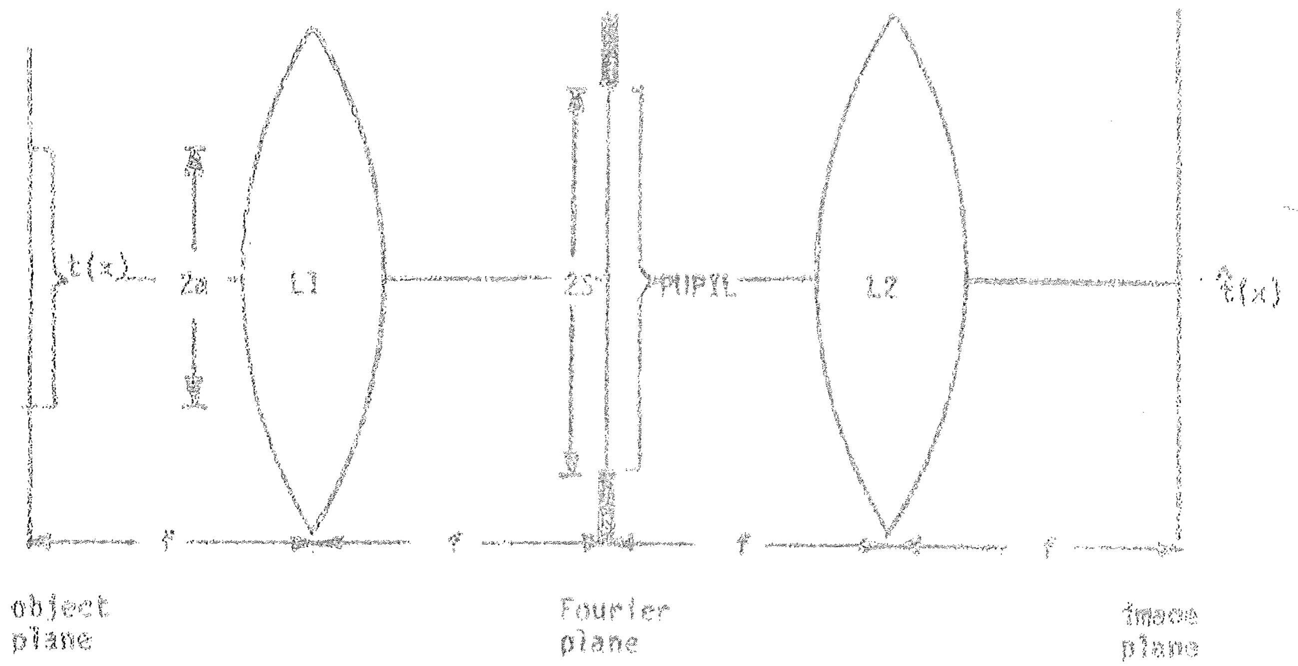


Fig.1: Geometry of a simple imaging system.

object, $t(x)$, is thus effectively placed through a low pass filter of bandwidth

$$2W = {}^{2S}/_{\lambda f} \tag{1}$$

where λ is the wavelength of the spatially coherent illumination and f is the focal length of L1. Disregarding coordinate reversal, the field amplitude on the back focal plane of L2 is

$$\hat{t}(x) = \int_{-W}^{W} T(v)\, e^{j2\pi v x}\, dv \tag{2}$$

where $T(v)$ is the Fourier transform of the object:

$$T(v) = \int_{-\infty}^{\infty} t(\xi)\, e^{-j2\pi v \xi}\, d\xi \tag{3}$$

Since the image is bandlimited, we may expand it via the sampling theorem:[4]

$$\hat{t}(x) = \sum_{n=-\infty}^{\infty} \hat{t}\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wx - n) \tag{4}$$

If the object is zero outside of the interval $x \in [-a, a]$, then to a good approximation, we may write Eq.4 as

$$\hat{t}(x) \simeq \sum_{n=-S/2}^{S/2} \hat{t}\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wx - n) \tag{5}$$

where the space-bandwidth product

$$S = 4Wa \tag{6}$$

is the required number of sample values to characterize $\hat{t}(x)$. Di Francia terms S the Shannon number, which is a measure of the amount of information or degrees of freedom capable of being transmitted by the given imaging system.

Di Francia concluded that, due to the finite capacity of the system, different objects might give rise to equivalent images. As noted by Wolter[5] and others, such is not the case for an object of finite support, since, incident on the rear focal plane of L1 (Fig.1) is a bandlimited function. Bandlimited functions have been shown to be analytic.[7] Any analytic function known over a finite neighborhood can be constructed exactly in the region of its analyticity. (For example, consider the region of convergence of a Taylor series expanded about a single analytic point.[8]) Thus, it follows that since we know $T(\nu)$ over the finite pupil in the Fourier plane, and since $T(\nu)$ is analytic, we know $T(\nu)$ everywhere. Thus, with knowledge of $\hat{t}(x)$, we know $t(x)$, and, as modeled, the imaging system has an infinite capacity for transmittance of information! One must furthermore conclude that, contrary to di Francia's statement, a given image must correspond to a unique object.

As we shall see, though, the above "something for nothing" statements are true only in the strictest of mathematical senses and disregard important physical realities, the foremost of which is noise.

The problem with superresolution

Consider Fig.2 in which a face and a corresponding nose are pictured. By previous arguments, if the face can be represented by a (two dimensional) analytic function, then, with knowledge of only the nose, we could, in principle, reconstruct the entire face! Such a use of analytic continuation is known in the optics community as superresolution.

Analytic continuation for the case where one has knowledge of the function over a finite (rectangular) region has been admiringly presented by Slepian and Pollak.[10] With reference again to Fig.1, we may express the image, $\hat{t}(x)$, as

$$\hat{t}(x) = \int_{-a}^{a} t(\xi)\, \mathrm{sinc}\, 2W(x-\xi)\, d\xi \tag{7}$$

The corresponding integral equation is

$$\lambda_n \psi_n(x) = \int_{-a}^{a} \psi_n(\xi)\, \mathrm{sinc}\, 2W(x-\xi)\, d\xi \tag{8}$$

Fig.2: A nose and a corresponding face

Solution of this relationship gives orthonormal eigen functions, $\psi_n(x)$ which are proportional to appropriately scaled spheroidal wave functions which have the unusual property of being orthonormal over the interval $[-a,a]$ and over $[-\infty,\infty]$. The λ_n are, of course, the resulting eigenvalues.

We may now expand the object in an orthonormal series:

$$t(x) = \sum_{n=-\infty}^{\infty} c_n \psi_n(x) \tag{9}$$

where

$$c_n = \int_{-\infty}^{\infty} \psi_n^*(x)\, t(x)\, dx \tag{10}$$

It follows from Eqs.7 and 8 that the corresponding image is simply

$$\hat{t}(x) = \sum_{n=-\infty}^{\infty} \lambda_n c_n \psi_n(x) \tag{11}$$

Thus, with knowledge of the image in terms of its coefficients, $\lambda_n c_n$, we have exact knowledge of the object simply by weighting each coefficient with its corresponding eigenvalue. This constitutes superresolution.

Slepian and Sonnenblick[11] have evaluated the eigenvalues of Eq.8 which are here roughly sketched in Fig.3. Note the step function nature of the curve, which, as noted by di Francia[12], goes nearly zero for n larger than the Shannon number. We must therefore conclude that information about the object in Eq.11 is almost completely carriedby the first ψ_n's up to n=S, while information carried by higher values of n is virtually lost.

This result reveals the weakness of superresolution. For $n > S$, small perturbations in $\lambda_n c_n$ due to noise or measurement error will cause drastic changes in the computed object. The idea of degrees of freedom, as proposed by di Francia, thus remains valid since only S output coefficients can be physically measured to a sufficient degree of

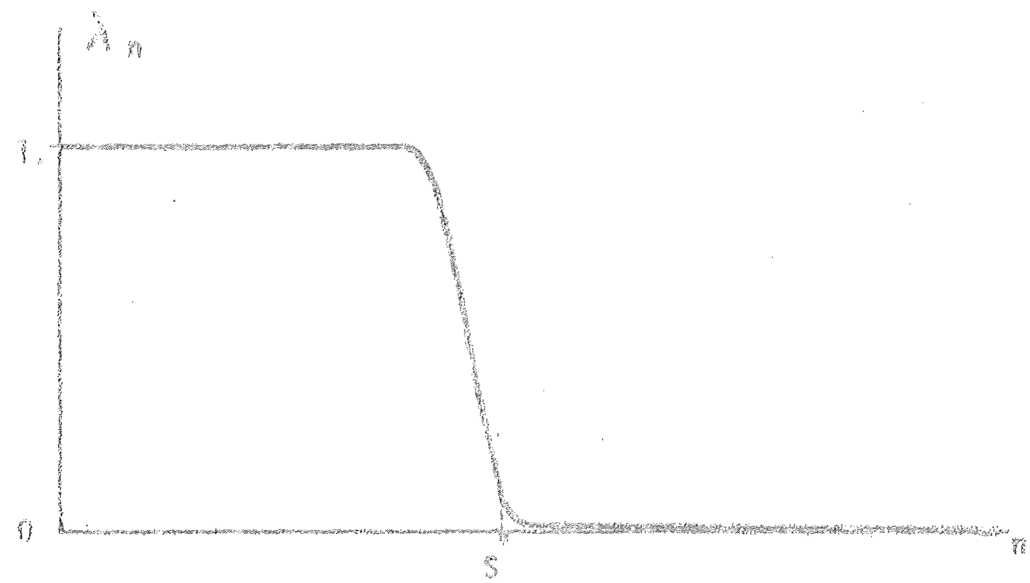


FIG 3: A sketch of the eigenvalues for the prolate spheroidal wave functions. (The curve has been smoothed by connecting adjacent points for clarity of presentation).

accuracy.

Attempts at implementing superresolution have been reported in the literature. Evans[13] presents an interesting scheme whereby apriori knowledge of the object is utilized. Sato, Ueda, and their colleagues[14] have presented schemes by which more information may be transmitted through the imaging system.

Effects of coherence on degrees of freedom

1. Di Francia vs. Walther

The first discussion of the effect of coherence on the information capacity of an imaging system must again be credited to di Francia.[12] He reasoned as follows. For coherent illumination, the DOF is as given in Eq.6. For incoherent illumination, one must essentially replace the pupil in Fig.7 with its autocorrelation which will have twice the spatial extent of the original pupil. Regarding only this spatial extent as a measure of bandwidth, we see that the number of DOF is almost doubled:

$$S_i \cong [2(2W)](2a) = 8aW \tag{12}$$

However, we have failed to take into account that S in Eq.6 denotes the number of complex samples, each of which represents two real samples. Thus, we must rewrite S as

$$S_c \approx 2(4aW) = 8aW \tag{13}$$

Since S_i is real, we must conclude that, in all practical cases, the number of DOF for this one dimensional example is equivalent for both coherent and incoherent illumination.

For certain pupils in two dimensions, di Francia claims this is not the case. For example, a (pseudo-one-dimensional) thin ring pupil of radius r transmits a number of DOF proportional to its arc length (and thus r) for the case of coherent illumination. This follows from the one dimensional nature. For incoherent illumination, we are interested in the pupil autocorrelation which covers the area of an entire disc. Again employing the argument of proportionality of pupil area and DOF, one concludes that, for incoherent illumination, the information capacity of the system is greatly increased. That is, the number of DOF is now proportional to r^2.

In the above argument, we have ignored the fact that for incoherent illumination there is a roll off in attenuation of higher frequencies. Refuting di Francia's observations, and defending one of his earlier papers, Walther[15] claims this omission invalidates di Francia's conclusions. In fact, the inability to claim information from the higher attenuated frequencies led Walther to conclude that the information capacity of the imaging system was equivalent for both coherent and incoherent illumination regardless of pupil geometry. In the same letter,[15] di Francia stands by his conclusion and cites experimental evidence in its support. Di Francia, as we shall see, was correct.

2. The pupil of points approach

The problem of the effects of partial coherence was addressed about one year after the di Francia-Walther conflict by Gori and Guattari.[16] A later paper by the same authors[17] is also worthy of noting.

Gori and Guattari considered the imaging system in Fig. 4 which is similarly analyzed by Goodman[18]. The image irradiance on the image plane is:

$$\mathcal{I}(x) = \iint_{-\infty}^{\infty} P(\xi_1)P^*(\xi_2)\, e^{j\frac{2\pi}{\lambda f}(\xi_1-\xi_2)x} \cdot \left\{\int_{-\infty}^{\infty} S(\sigma)\, t(\sigma+\xi_1)\, t^*(\sigma-\xi_2)\, d\sigma\right\} d\xi_1\, d\xi_2 \qquad (14)$$

where

$P(\xi)$ is the pupil function on the pupil plane.

$S(\sigma)$ is the quasimonochromatic radiance distribution on the source plane.

$t(y)$ is the complex amplitude transmittance on the object plane.

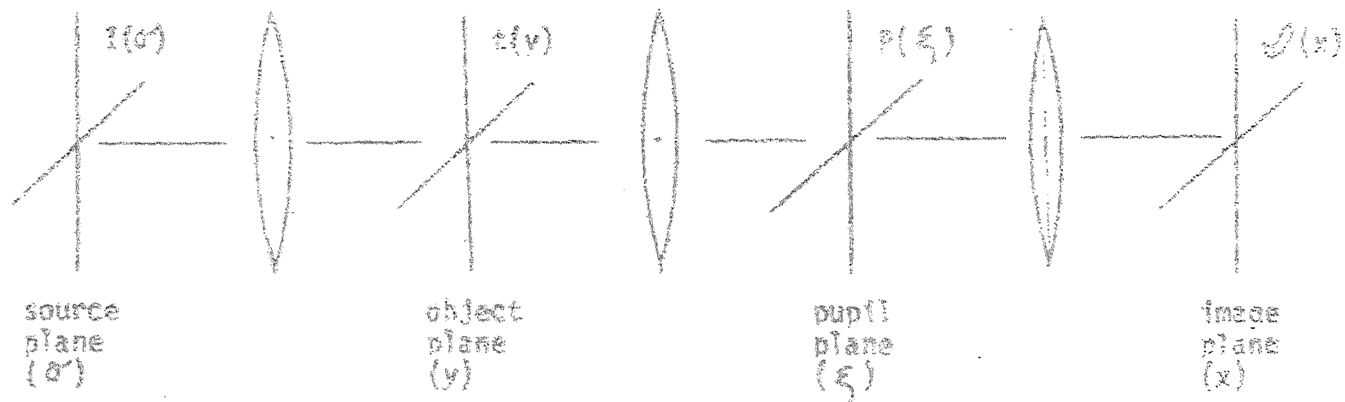


Fig.4: Imaging system geometry for the purpose of determining the effect of the source's degree of coherance on the capability of the system to transmit information.

We consider the special case where the pupil is an array of point-like sources:

$$P(\xi) = \sum_{\alpha=1}^{n} P_\alpha \, \delta(\xi - \xi_\alpha) \tag{15}$$

Here, P_α and ξ_α are the respective amplitude and location of these points and $\delta(x)$ is the Dirac delta. Substituting into Eq.14 and simplifying gives

$$\mathcal{D}(x) = \mathcal{D}_\alpha + 2 \sum_{\alpha>\beta}^{n} \mathcal{D}_{\alpha,\beta} \cos\left\{\frac{2\pi}{\lambda f}(\xi_\alpha - \xi_\beta)x + \phi_{\alpha,\beta}\right\} \tag{16}$$

where

$$\mathcal{D}_\alpha = \sum_{\alpha=1}^{n} |P_\alpha|^2 \int_{-\infty}^{\infty} I(\sigma)\,|t(\sigma+\xi_\alpha)|^2 \, d\sigma \tag{17}$$

and

$$\mathcal{D}_{\alpha,\beta}\, e^{i\phi_{\alpha,\beta}}\Big|_{\alpha\neq\beta} = P_\alpha P_\beta^* \times \int_{-\infty}^{\infty} I(\sigma)\, t(\sigma+\xi_\alpha)\, t^*(\sigma+\xi_\beta)\, d\sigma \tag{18}$$

From these relationships, we see that, in order to express $\mathcal{D}(x)$, we need knowledge of $n(n-1)$ terms. Specifically, $n(n-1)/2$ phase terms,

$\phi_{\alpha,\beta}$, $n(n-1)/2$ intensity terms, $\mathcal{D}_{\alpha,\beta}$, and lastly, the single value $\mathcal{D}_{\alpha}$. Thus, assuming these terms are independent and unequal, we have a number of DOF equal to

$$N_L = n(n-1) + 1 \tag{19}$$

However, some terms might be dependent or equal. Thus, N_L must be the upper bound on the number of DOF for the given system.

To examine the possibility of equivalence of terms, Gori and Guattari introduce the geometric-efficiency factor η. Some reflection on the reader's part will reveal that the number of unique values in Eq.16 is the number of points, N_{max}, in the pupil's autocorrelation function. As far as the pupil is concerned, N_{max} is the maximum allowable number of DOF. The geometric-efficiency factor for the pupil is thus defined as

$$\eta = N_{max} / N_L \tag{20}$$

It follows that, for a given pupil, ηN_L real parameters are needed to define $\mathcal{D}(x)$ in Eq.16. It remains to determine the corresponding information required to specify $\mathcal{D}(x)$. That is, what dependence exists between the various parameters.

We consider first the limiting cases of coherent and incoherent illumination. For the coherent case, we may write

$$I(\sigma) = \delta(\sigma) \tag{21}$$

Equations 17 and 18 thus become

$$\mathcal{J}_{\alpha} = \sum_{\alpha=1}^{n} | P_{\alpha} t(\xi_{\alpha}) |^2 \tag{22}$$

and

$$\begin{aligned} \mathcal{J}_{\alpha,\beta}\, e^{j\phi_{\alpha,\beta}} \Big|_{\alpha\neq\beta} &= |P_{\alpha} t(\xi_{\alpha})|\, |P_{\beta} t(\xi_{\beta})| \\ &\times e^{j[\arg P_{\alpha} t(\xi_{\alpha}) - \arg P_{\beta} t(\xi_{\beta})]} \end{aligned} \tag{23}$$

If we know $|t|$ and all of the differences $\arg P_{\alpha} t(\xi_{\alpha}) - \arg P_{\beta} t(\xi_{\beta})$, then we have complete knowledge of $\mathcal{J}(x)$ in Eq. 16. This constitutes $\overset{2n-1}{n(n-1)}$ degrees of freedom.

For the incoherent case

$$I(\sigma) = I_0 \tag{24}$$

Equations 17 and 18 become

$$\mathcal{J}_{\alpha} = \sum_{\alpha=1}^{n} |P_{\alpha}|^2 I_0 \int_{-\infty}^{\infty} |t(\sigma)|^2 d\sigma \tag{25}$$

and

$$\begin{aligned} \mathcal{J}_{\alpha,\beta}\, e^{j\phi_{\alpha,\beta}} \Big|_{\alpha\neq\beta} &= P_{\alpha} P_{\beta} I_0 \\ &\times \int_{-\infty}^{\infty} t(\sigma)\, t^{*}(\sigma - \xi_{\alpha} - \xi_{\beta})\, d\sigma \end{aligned} \tag{26}$$

Both Eq.25 and Eq.26 are determined by the $N_{max} = M N_L$ terms in the auto-correlation of t. Thus, we have N_{max} DOF and have reached the previously derived upper bound.

For the partially coherent case, we expect the number of DOF to be somewhere between the worst case coherent and optimal incoherent number. We model our partially coherent source as a number of point elements:

$$I(\sigma) = \sum_{\gamma=1}^{m} I_\gamma \, \delta(\sigma - \sigma_\gamma) \tag{27}$$

Equations 17 and 18 become

$$\mathcal{D}_\alpha = \sum_{\alpha=1}^{n} |P_\alpha|^2 \sum_{\gamma=1}^{m} I_\gamma \, |t(\sigma_\gamma + \xi_\alpha)|^2 \tag{28}$$

and

$$\mathcal{D}_{\alpha,\beta} \, e^{j\phi_{\alpha,\beta}}\Big|_{\alpha \neq \beta} = P_\alpha P_\beta \sum_{\gamma=1}^{m} I_\gamma \, |t(\sigma_\gamma + \xi_\alpha)| \, |t(\sigma_\gamma + \xi_\beta)| \times e^{j[\arg t(\sigma_\gamma + \xi_\alpha) - \arg t(\sigma_\gamma + \xi_\beta)]} \tag{29}$$

If we want to express all of the terms in Eqs.28 and 29, by knowledge of the complex function t, we should know $|t|$ at all of the points $\sigma_\gamma + \xi_\alpha$, and the differences of arg(t) at these same points. To evaluate the number of distinct values of the sum $\sigma_\gamma + \xi_\alpha$, we must count the number M of the points of the convolution function between the source radiance distribution and the pupil function. This amounts to $2M-1$ real data. Ergo, the $\mathcal{n} M_L$ unknown terms are all independent only if $2M-1 \geq \mathcal{n} M_L$. Otherwise, only $2M-1$ terms are independent. This is the resulting number of DOF. Employing the coherent limit as the lower bound, the minimum value of the DOF is $2n-1$.

Discussion

The DOF measure of the information capacity of an imaging system as introduced by di Francia has withstood attacks of superresolution principles. The number of DOF has been shown to increase monotonically with the degree of incoherence of illumination for the case of point like pupils by Gori and Guattari. This is in direct counterdiction with Walther's statement that DOF was independent of the degree of coherence.

It is this author's conjecture that sampling theorem notions could be applied to characterize the effects of coherence for a larger class of pupils. This follows from the investigation of Gori and Guattari where the 1) pupil 2) input transmittance and 3) source were all modeled as sampled processes at some time in the analysis.

Awknowledgements

The author wishes to express his deep appreciation to himself for gathering and reading appropriate material as well as for writing this paper. Thanks go also to Bob Marks who typed the manuscript and drew the figures.

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INTRODUCTION TO STATISTICAL OPTICS

by J. W. GOODMAN

VOCABULARY AND DEFINITIONS

HUYGEN'S FRESNEL PRINCIPLE: $U(P_0,\nu) = \frac{1}{j\lambda}\int\int_{\Sigma} U(P,\nu)\frac{e^{j2\pi r/\lambda}}{r}\chi(\theta)ds$

OBLIQUITY FACTOR

CIRCULAR COMPLEX RANDOM PROCESS

THERMAL LIGHT

DEGREE OF POLARIZATION: $P = \frac{I_1 - I_2}{I_1 + I_2}$

STRUCTURE FUNCTION: $D_\theta \triangleq E\left[(\theta(t_2) - \theta(t_1))^2\right]$

MICHELSON INTERFEROMETER

TEMPORAL COHERENCE

INTERFEROGRAM

SELF-COHERENCE FUNCTION: $\Gamma(\tau) = \langle u(t+\tau)\, u^*(t)\rangle$

COMPLEX DEGREE OF COHERENCE: $\gamma(\tau) = \Gamma(\tau)/\Gamma(0)$

FRINGE VISIBILITY: $\mathcal{V} = I_{MAX} - I_{MIN} / I_{MAX} + I_{MIN}$

COHERENCE TIME: $\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau \approx \frac{1}{\Delta\nu}$

POWER SPECTRAL DENSITY: $\Gamma(\tau) = \int_0^\infty 4\,\mathcal{G}^{(r,r)}(\nu)\, e^{-j2\pi\nu\tau} d\nu$

NORMALIZED POWER SPECTRAL DENSITY: $\hat{\mathcal{G}}(\nu) = \mathcal{G}^{(r,r)}(\nu) / \int_0^\infty \mathcal{G}(\nu) d\nu$

FOURIER SPECTROSCOPY

YOUNG'S EXPERIMENT

MUTUAL COHERENCE FUNCTION: $\Gamma_{12}(\tau) = \langle u_1(t+\tau)\, u_2^*(t)\rangle$

COMPLEX COHERENCE FACTOR: $\gamma_{12}(\tau) = \Gamma_{12}(\tau) / \sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}$

SPATIAL COHERENCE: $\gamma_{12}(0)$

PARAXIAL APPROXIMATION: $r_2 - r_1 \approx \frac{1}{2z_2}\left[\rho_2^2 - \rho_1^2 + 2\Delta\xi x + 2\Delta\eta y\right]$

NARROWBAND ASSUMPTION: $\Delta\nu << \bar{\nu}$

FRINGE WASHOUT

COHERENCE LENGTH ASSUMPTION: $r_2 - r_1 << c\tau_c$

MUTUAL INTENSITY: $J_{12} = \Gamma_{12}(0) = \langle u(P_1,t)\, u^*(P_2,t)\rangle$

COMPLEX COHERENCE FACTOR: $\mu_{12} = J_{12} / \sqrt{I_1 I_2}$

AIRY PATTERN

FRAUNHOFFER DIFFRACTION

QUASIMONOCHROMATIC ASSUMTION

LAPLACIAN OPERATOR: $\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$

HILBERT TRANSFORM: $f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{F(\xi)}{\xi - x} d\xi$

HELMHOLTZ EQUATION: $\nabla^2 J_{12} + \bar{k}^2 J_{12} = 0$

SCHWARZ'' INEQUALITY

EVANESCENT WAVES

VAN CITTERT - ZERNIKE THEOREM

COHERENCE AREA: $A_c \triangleq \int_{-\infty}^{\infty}\int |\mu(\Delta x, \Delta y)|^2 d\Delta x \, d\Delta y$

RAYLEIGH DISTANCE

AMBIGUITY FUNCTION

FIZEAU STELLAR INTERFEROMETER

ANGULAR DIAMETER

MICHELSON STELLAR INTERFEROMETER

ENSEMBLE AVERAGE

OPTICAL TRANSFER FUNCTION (OTF)

POINT-SPREAD FUNCTION

HOMOGENEOUS

ISOTROPIC

TURBULONS

INNER AND OUTER SCALES

STRUCTURE CONSTANT: C_N^2

RYTOV APPROXIMATION

LOG AMPLITUDE

LOG NORMAL DISTRIBUTION

AUTOCOVARIANCE: $\tilde{\Gamma}_\chi(\Delta_x, \Delta_y) = E[\{\chi(x,y) - \bar{\chi}\}\{\tilde{\chi}(x-\Delta_x, x-\Delta_y) - \bar{\tilde{\chi}}\}]$

SCINTILLATIONS

MODULATION TRANSFER FUNCTION (MTF)

POISSON DISTRIBUTION: $p(K; t; \tau/W) = \frac{[\alpha W_r(\tau)]^K}{K!} e^{-\alpha W_r(\tau)}$

QUANTUM EFFICIENCY: η

MANDEL'S FORMULA: $p(K,\tau) = \int_0^\infty \frac{(\alpha W)^K}{K!} e^{-\alpha W} p_W(W) dW$

BOSE-EINSTEIN DISTRIBUTION: $p(K,\tau) = \frac{1}{1+\bar{K}} \left(\frac{\bar{K}}{1+\bar{K}}\right)^K$

CORRELATION INTERVAL

BOXCAR APPROXIMATION

GAMMA DISTRIBUTION: $p_W(W) = \left(\frac{m}{\bar{W}}\right)^m \frac{1}{\Gamma(m)} W^{m-1} e^{-mW/\bar{W}} \mu(W)$

NEGATIVE BINOMIAL DISTRIBUTION

DEGENERACY PARAMETER: $\delta_c = \bar{K}/m$

SHOT NOISE

WAVE DEGENERACY PARAMETER: $\delta_w = \delta_c/n$

CROSS SPECTRALLY PURE

BLACKBODY RADIATION

DISCRETE FOURIER TRANSFORM: $X(p_0) = \sum_0^{N-1} \frac{1}{N} K(\ell) e^{j \frac{2\pi}{N} \ell l}$

INTENSITY INTERFEROMETER

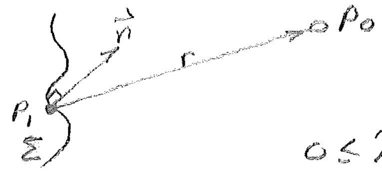
I. SOME FIRST ORDER PROPERTIES OF LIGHT BEAMS

A. PROPERTIES OF LIGHT WAVES

1. MONOCHROMATIC LIGHT

$$\underline{u}(P,t) = \underline{U}(P,\nu)\, e^{-j2\pi\nu t}$$

$\underline{U}(P,\nu)$ = PHASOR.



HUYGEN'S - FRESNEL PRINCIPLE:

$$\underline{U}(P_0,\nu) = \frac{1}{j\lambda}\iint_{\Sigma} \underline{U}(P_1,\nu)\, \frac{e^{j2\pi r/\lambda}}{r}\, \chi(\theta)\, ds$$

$0 \le \chi(\theta) \le 1$ = OBLIQUITY FACTOR $(r \gg \lambda)$

2. NON-MONOCHROMATIC LIGHT

WE MUST TRUNCATE $\underline{u}(P,t)$ TO $-T$ TO T. CALL IN $u_T(P,t)$

LET $\underline{u}_T(t)$ BE ANALYTIC SIGNAL ASSOCIATED WITH $u_T(P,t)$

$$\underline{u}_T(P,t) = \int_0^{\infty} 2\, \underline{U}_T(P,\nu)\, e^{-j2\pi\nu t}\, d\nu$$

WHERE $\underline{U}_T(P,\nu) = \lim_{T\to\infty} \int_{-T}^{T} u_T(P,t)\, e^{j2\pi\nu t}\, dt$

GOING THROUGH SOME MATH GIVES HUYGEN'S FRESNEL AS

$$\underline{u}(P_0,t) = \iint_{\Sigma} \frac{\frac{d}{dt}\, u(P_1, t - r/c)}{2\pi r c}\, \chi(\theta)\, ds$$

3. NARROWBAND LIGHT $(\nabla\nu \ll \bar{\nu})$

$$\underline{u}(P_0,t) \doteq \iint_{\Sigma} \frac{1}{j\lambda r}\, \underline{u}\left(P_1, t - \frac{r}{c}\right) \chi(\theta)\, ds$$

B. THERMAL LIGHT

1. POLARIZED THERMAL LIGHT

$u_x(P,t)$ = X COMPONENT OF POLARIZED FIELD

$u_i(P,t)$ = CONTRIBUTION FROM i^{TH} ATOM

$$\underline{u}_x(P,t) = \sum_{\text{ALL ATOMS}} \underline{u}_i(P,t)$$

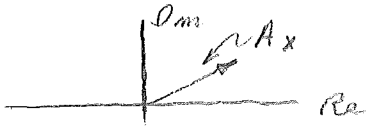
$$\underline{A}_x(P,t) = \sum_{\text{ALL ATOMS}} \underline{A}_i(P,t)$$

WHERE

$$\underline{A}_x(P,t) = \underline{u}_x(P,t)\, e^{j2\pi\bar{\nu}t}$$

$$\underline{A}_i(P,t) = \underline{u}_i(P,t)\, e^{j2\pi\bar{\nu}t}$$

BOTH $\underline{A}_x$ & $\underline{u}_x$ ARE <u>CIRCULAR COMPLEX GAUSSIAN RANDOM PROCESSES</u>



IF $\mathrm{Im}\,\underline{A}_x$ & $\mathrm{Re}\,\underline{A}_x$ ARE GAUSSIAN RANDOM PROCESSES WITH VARIANCES σ^2 AND $A \triangleq |\underline{A}_x(P,t)|$, THEN

$$p_A(A) = \frac{A}{\sigma^2}\, e^{-A^2/2\sigma^2}\, \mu(A) \Leftarrow \underline{\text{RAILEIGH DISTRIBUTION}}$$

FOR $I = A^2$, $\sigma_I = \bar{I} = 2\sigma^2$

$$p_I(I) = \frac{1}{\bar{I}}\, e^{-I/\bar{I}}\, \mu(I) \Leftarrow \underline{\text{EXPONENTIAL DISTRIBUTION}}$$

2. UNPOLARIZED THERMAL LIGHT

- TWO NECESSARY CONDITIONS:

  a. LIGHT PASSED BY POLARIZAR IS INDEPENT ON ORIENTATION

  b. FOR $\underline{U}_x$ AND $\underline{U}_y$ ORTHOGONAL FIELD COMPONENTS

$$\langle \underline{U}_x^*(t+\tau)\,\underline{U}_y(t)\rangle = 0 \quad \forall\, \tau$$

BOTH $\underline{U}_x$ & $\underline{U}_y$ ARE COMPLEX GAUSSIAN PROCESSES

$$\begin{aligned} I(P,t) &= |\underline{U}_x(P,t)|^2 + |\underline{U}_y(P,t)|^2 \\ &= |\underline{A}_x(P,t)|^2 + |\underline{A}_y(P,t)|^2 \\ &= I_x(p,t) + I_y(p,t) \end{aligned}$$

$$\bar{I}(p) = 2\bar{I}_x(p) = 2\bar{I}(p)$$

$$p_{I_x}(I_x) = \frac{2}{\bar{I}} e^{-2I_x/\bar{I}} \quad ; \quad p_{I_y}(I_y) = \frac{2}{\bar{I}} e^{-2I_y/\bar{I}}$$

$$\begin{aligned} p_I(I) &= p_{I_x}(I_x) * p_{I_y}(I_y) \\ &= (2/\bar{I})^2\, I\, e^{-2(I/\bar{I})}\, \mu(I) \end{aligned}$$

3. PARTIALLY POLARIZED THERMAL LIGHT

MAY ALWAYS EXPRESS I AS SUM OF TWO UNCORRELATED INTENSITY COMPONENTS:

$$I(P,t) = I_1(P,t) + I_2(P,t)$$

FOR GAUSSIAN, UNCORRELATED $\Rightarrow$ INDEPENDENT

<u>DEGREE OF POLARIZATION</u>

$$\mathcal{P} = \bar{I}_1 - \bar{I}_2 \,/\, \bar{I}_1 + \bar{I}_2 \qquad (\bar{I}_1 \geq \bar{I}_2)$$

THEN: $\bar{I}_1(P) = \frac{1}{2}(1+\mathcal{P})\bar{I} \qquad \bar{I}_2(P) = \frac{1}{2}(1-\mathcal{P})\bar{I}$

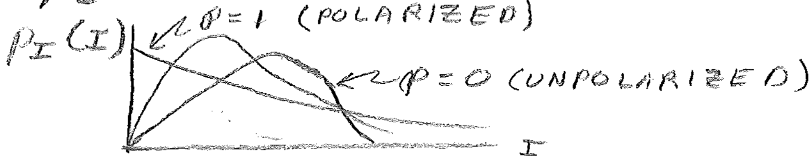
$$p_{I_1}(I_1) = \frac{2}{(1+\mathcal{P})\bar{I}} e^{\frac{-2I_1}{(1+\mathcal{P})\bar{I}}} \quad ; \quad p_{I_2}(I_2) = \frac{2}{(1-\mathcal{P})\bar{I}} e^{-2I_2/(1-\mathcal{P})\bar{I}}$$

THE CHARACTERISTIC FUNCTION OF I IS THE PRODUCT OF THE CHARACTERISTIC FUNCTIONS OF I_1 & I_2:

$$\begin{aligned} M_I(\omega) &= \left[\frac{1}{1 - j\frac{\omega}{2}(1+\mathcal{P})\bar{I}}\right]\left[\frac{1}{1 - j\frac{\omega}{2}(1-\mathcal{P})\bar{I}}\right] \\ &= \frac{(1+\mathcal{P})/2\mathcal{P}}{1 - j\,\omega/2\,(1+\mathcal{P})\bar{I}} - \frac{(1-\mathcal{P})/2\mathcal{P}}{1 - j\,\omega/2\,(1-\mathcal{P})\bar{I}} \end{aligned}$$

THE RESULTING DENSITY IS

$$p_I(I) = \frac{1}{\mathcal{P}\bar{I}}\left[e^{-\frac{2I}{(1+\mathcal{P})\bar{I}}} - e^{-\frac{2I}{(1-\mathcal{P})\bar{I}}}\right]$$



C. LASER LIGHT

1. SINGLE MODE OSCILLATION (HIGHLY IDEALIZED)

a. $u(t) = S \cos[2\pi\nu_0 t - \phi]$

S AND ν_0 KNOWN. LINEAR POLARIZATION ASSUMED.

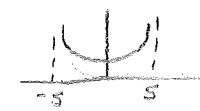
ϕ IS A RANDOM VARIABLE WITH UNIFORM DISTRIBUTION ON $-\pi, \pi$

CONSIDER $u(0) = S\cos\phi$

CHARACTERISTIC FUNCTION OF U IS

$M_U(\omega) = E[e^{j\omega S\cos\phi}] = J_0(\omega S)$

$\Rightarrow p_U(u) = \frac{1}{\pi\sqrt{S^2+u^2}}\, \mathrm{rect}\left[\frac{u}{2S}\right]$



THE INTENSITY IS $I = S^2$

$p_I(I) = \delta(I - S^2)$

b. $u(t) = S\cos\{2\pi\nu_0 t - \Theta(t)\} = S\cos\psi(t)$

$\nu_i(t) =$ INSTANTANEOUS FREQUENCY $\triangleq \frac{1}{2\pi}\frac{d\psi}{dt} = \nu_0 - \frac{1}{2\pi}\frac{d\Theta}{dt}$

"A-C" COMPONENT OF FREQUENCY: $\nu_R(t) = \frac{1}{2\pi}\frac{d\Theta}{dt}$

$\nu_R(t)$ IS ZERO-MEAN & STATIONARY

$\Theta(t) = 2\pi\int_{-\infty}^{t}\nu_R(\xi)d\xi$ IS NON-STATIONARY.

(STRUCTURE FUNCTION, THOUGH, IS IND. OF ORIGIN)

$D_\Theta(t_1,t_2) \triangleq \overline{[\Theta(t_2) - \Theta(t_1)]^2} \Leftarrow$ STRUCTURE FUNCTION

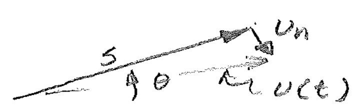
$= 4\pi^2\left[\int_{-\infty}^{\infty}\mathrm{rect}\left[\frac{\xi-(t_1+t_2)/2}{t_2-t_1}\right]\nu_R(\xi)\,d\xi\right]^2$

$= 4\pi\tau\int_{-\infty}^{\infty}\Lambda(\eta/\tau)\Gamma_\nu(\eta)\,d\eta$

$\simeq 4\pi^2\tau\int_{-\infty}^{\infty}\Gamma_\nu(\eta)\,d\eta$ FOR $\tau >>$ CORRELATION TIME

c. $u(t) = S\cos[2\pi\nu_0 t - \Theta(t)] + u_n(t)$

$u_n(t)$ IS GAUSSIAN & INDEPENDENT OF $\Theta(t)$



$|u_n(t)| = \underline{A}_n(t)$

$I = |\underline{S} + \underline{A}_n|^2 = |\underline{S}|^2 + 2\,\mathrm{Re}[\underline{S}^*\underline{A}_n]$

LET $\underline{S} = S e^{j\theta}$, $\underline{A}_n = A_n e^{j\phi_n}$

$\mathrm{Re}\, S^* A_n$ IS GAUSSIAN WITH $\sigma_I^2 = 4S^2\overline{A_n^2}\cos^2(\theta - \phi_n) = 2 I_S \overline{I_N}$

$p_I(I) \simeq \frac{1}{\sqrt{4\pi I_S \overline{I_N}}}\, e^{-(I-I_S)^2/4I_S\overline{I_N}}$; $I_S >> \overline{I_N}$

d• RISKEN'S MODEL

$$p_I(I) = \frac{2}{\pi I_0}\,\frac{1}{1+\mathrm{erf}\,w}\, e^{-\left(\frac{I}{\sqrt{\pi} I_0} - w\right)^2} \mu(I)$$

I_0 = AVERAGE INTENSITY AT THRESHOLD

W = PARAMETER

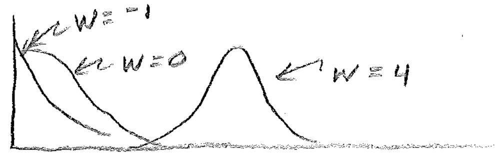
$$\bar{I} = I_0\left[\sqrt{\pi}\, w + e^{-w^2} / 1 + \mathrm{erf}\, w\right.$$

$$\mathrm{erf}\, W \triangleq \frac{2}{\sqrt{\pi}} \int_0^W e^{-x^2} dx \Leftarrow \underline{\text{ERROR FUNCTION}}$$

$W \ll 0 \Rightarrow p_I(I) = \frac{1}{\pi I_0} e^{-w^2} e^{-\frac{2w}{\sqrt{\pi} I_0} I} \mu(I) \Leftarrow$ THERMAL

$W = 0 \Rightarrow p_I(I) = \frac{2}{\pi I_0} e^{-I^2/\pi I_0^2} \mu(I) \Leftarrow \frac{1}{2}$ OF GAUSSIAN

$W \gg 0 \Rightarrow p_I(I) = \frac{1}{\pi I_0} e^{-\left(I - w\sqrt{\pi} I_0 / \sqrt{\pi} I_0\right)^2} \mu(I) \Leftarrow$ GAUSSIAN



2• MULTIMODE LASER LIGHT

$$u(t) = \sum_{i=1}^{N} S_i \cos\{2\pi \nu_i t - \theta_i(t)\}$$

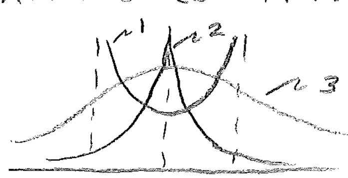
ASSUME THE VARIOUS MODES ARE INDEPENDENT.

FOR A SINGLE MODE (AS BEFORE) $M_i(\omega) = J_0(\omega S_i)$

FOR N MODES OF EQUAL AMPLITUDES:

$$M_u(\omega) = J_0^N\left(\omega \sqrt{I/N}\right)$$

PERFORMING (DIGITAL) FOURIER TRANSFORM:



ALSO WE MAY SHOW

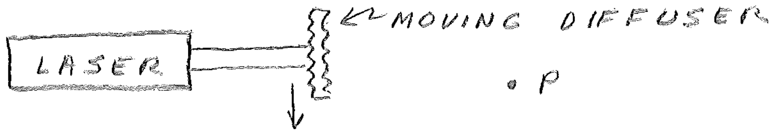
$$\sigma_N / \bar{I} = \left[\left(\frac{N-1}{N}\right)^2 \left(\frac{\sigma_{N-1}}{\bar{I}}\right)^2 + \left(\frac{N-1}{N}\right)^2 + \frac{1}{N^2} + \frac{4(N-1)}{N^2} - 1\right]^{\frac{1}{2}}$$

$\sigma_1 = 0$

FOR $n > 5$, WE HAVE, FOR ALL PRACTICAL PURPOSES, THERMAL LIGHT.

• 3 • QUASI-THERMAL LIGHT PRODUCED BY PASSING LASER LIGHT THROUGH A MOVING DIFFUSER

WE MAY GENERATE "THERMAL" LIGHT WITH LASER

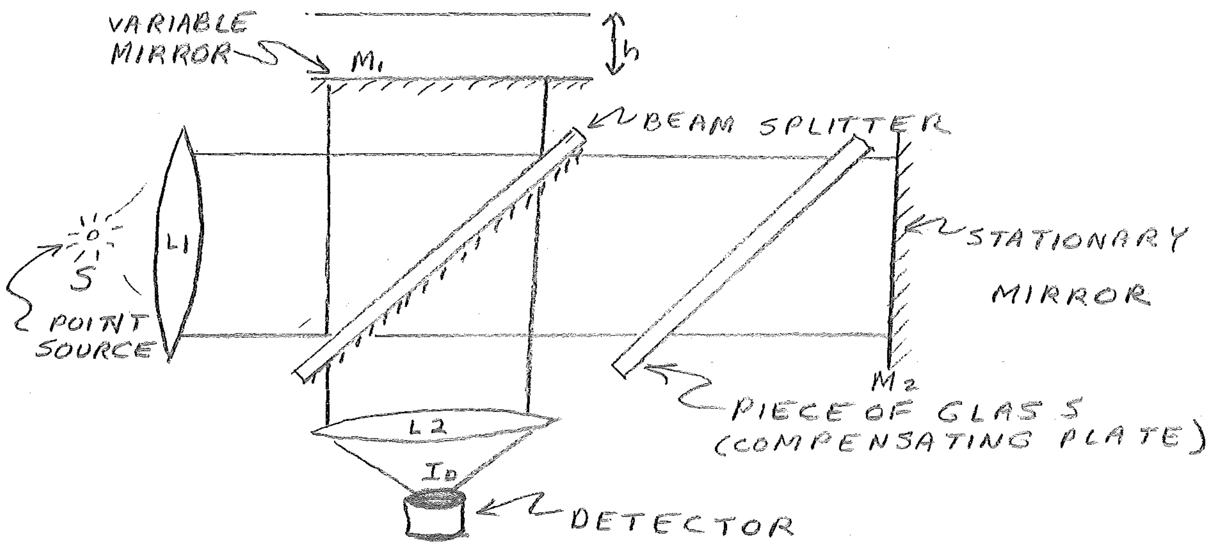


THE COMPLEX FIELD OBSERVED @ P MAY BE THOUGHT OF AS CIRCULAR GAUSSIAN RANDOM PROCESS

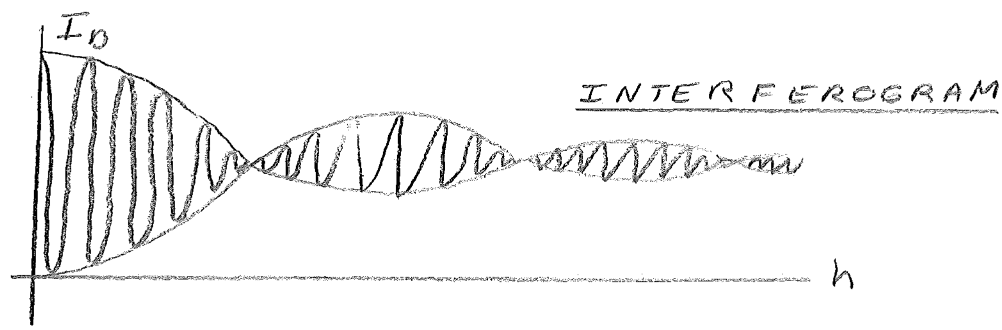
II. THEORY OF SECOND ORDER COHERENCE

A. TEMPORAL COHERENCE

1. THE MICHELSON INTERFEROMETER



- COMPENSATING PLATE USED SO THAT BOTH BEAMS TRAVEL THE SAME DISTANCE IN "GLASS"
- WILL GET DESTRUCTIVE/CONSTRUCTIVE INTERFERENCE AS M1 MOVES. ENVELOPE OF THIS SINUSOIDAL VARIATIONS IS WHAT IS OF INTEREST.



• 2 • MATHEMATICAL DESCRIPTION OF THE EXPERIMENT

THE LIGHT INTENSITY FALLING ON THE DETECTOR IS

$$I_D(h) = \langle |K_1 U(t) + K_2 U(t + \tfrac{2h}{c})|^2 \rangle$$
$$= K_1^2 \langle |U(t)|^2 \rangle + K_2^2 \langle |U(t + \tfrac{2h}{c})|^2 \rangle$$
$$+ K_1K_2 \langle U(t + \tfrac{2h}{c}) U^*(t) \rangle + K_1K_2 \langle U^*(t + \tfrac{2h}{c}) U(t) \rangle$$

WHERE K_1 & K_2 DENOTE LOSSES IN RESPECTIVE BEAMS

$$I_0 = \langle |U(t)|^2 \rangle = \langle |U(t + \tfrac{2h}{c})|^2 \rangle$$

$\underline{\Gamma}(\tau) = \langle U(t+\tau) U^*(t) \rangle \Rightarrow$ SELF COHERENCE FUNCTION

$$\Rightarrow I_D(h) = (K_1^2 + K_2^2) + 2K_1K_2 \, Re \, \underline{\Gamma}(\tfrac{2h}{c})$$

NOTING THAT $\underline{\Gamma}(0) = I_0$,

$\underline{\gamma}(\tau) \triangleq \underline{\Gamma}(\tau)/\underline{\Gamma}(0) \Rightarrow$ COMPLEX DEGREE OF COHERENCE

$$\gamma(0) = 1 \qquad 0 \le |\underline{\gamma}(\tau)| \le 1$$

$$\Rightarrow I_D(h) = (K_1^2 + K_2^2) I_0 \left[1 + \frac{2K_1K_2}{K_1^2 + K_2^2} Re \, \underline{\gamma}(\tfrac{2h}{c}) \right]$$

LET $\underline{\gamma}(\tau) = \gamma(\tau) \exp\left[-j \{ 2\pi \bar{\nu} \tau - \alpha(\tau) \} \right]; \; \tau = \frac{2h}{c}$

AND ASSUME EQUAL LOSSES: $K = K_1 = K_2$

$$\Rightarrow I_D(h) = 2K^2 I_0 \{ 1 + \gamma(\tfrac{2h}{c}) \cos [2\pi\bar{\nu}\tau - \alpha(\tfrac{2h}{c})] \}$$

$\mathcal{V} \triangleq I_{MAX} - I_{MIN} / I_{MAX} + I_{MIN} \Rightarrow$ FRINGE VISIBILITY

$$\mathcal{V}(h) = |\underline{\gamma}(2h/c)| = \gamma(2h/c)$$

FOR $K_1 \neq K_2 \Rightarrow \mathcal{V}(h) = \frac{2K_1K_2}{K_1^2 + K_2^2} \gamma(\frac{2h}{c})$

WHEN THE VISIBILITY ESSENTIALLY GOES TO ZERO, WE HAVE EXCEEDED THE COHERENCE LENGTH OR, EQUIVALENTLY, THE COHERENCE TIME.

THE CONCEPT OF COHERENCE HAS TO DO WITH THE ABILITY OF LIGHT BEAMS TO FORM FRINGES.

• 3 • RELATION BETWEEN THE INTERFEROGRAM AND THE POWER SPECTRAL DENSITY OF THE LIGHT

$$\underline{\Gamma}(\tau) = \int_0^\infty 4\, \mathcal{G}^{(r,r)}(\nu)\, e^{-j2\pi\nu\tau}\, d\nu$$

$\mathcal{G}^{(r,r)}(\nu)$ = <u>POWER</u> <u>SPECTRAL</u> <u>DENSITY</u> OF $u(t)$

$$\underline{\gamma}(\tau) = \int_0^\infty 4\, \mathcal{G}^{(r,r)}(\nu)\, e^{-j2\pi\nu\tau}\, d\nu \Big/ 4\int_0^\infty \mathcal{G}^{(r,r)}(\nu)\, d\nu$$

$$= \int_0^\infty \hat{\mathcal{G}}(\nu)\, e^{-j2\pi\nu\tau}\, d\nu$$

$$\hat{\mathcal{G}}(\nu) = \mathcal{G}^{(r,r)}(\nu) \Big/ \int_0^\infty \mathcal{G}^{(r,r)}(\nu)\, d\nu$$ ← <u>NORMALIZED</u> <u>PWR. SPEC. DEN.</u>

$$\int_0^\infty \hat{\mathcal{G}}(\nu)\, d\nu = 1$$

VARIOUS LIGHT TYPES:

a. LOW-PRESSURE ⇒ $\hat{\mathcal{G}}(\nu) \doteq \frac{2\ln 2}{\sqrt{\pi}\,\Delta\nu}\, e^{-\left(2\ln 2\, \frac{\nu-\bar{\nu}}{\Delta\nu}\right)^2}$ ← GAUSSIAN

$$\underline{\gamma}(\tau) = \exp -\left(\frac{\pi\Delta\nu\tau}{2\sqrt{\ln 2}}\right)^2 e^{-j2\pi\bar{\nu}\tau}$$

$$|\underline{\gamma}(\tau)| = \gamma(\tau) = \exp\left[-\left(\frac{\pi\Delta\nu\tau}{2\sqrt{\ln 2}}\right)^2\right]$$

b. HIGH-PRESSURE ⇒ $\frac{2}{\pi\Delta\nu}\; \frac{1}{1+\left(2\frac{\nu-\bar{\nu}}{\Delta\nu}\right)^2}$ ← <u>LORENTZIAN</u>

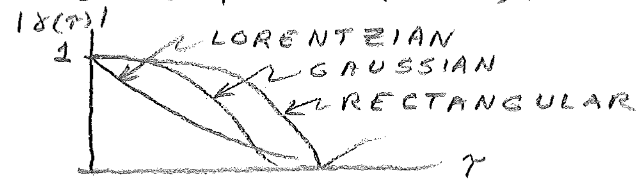
$$\underline{\gamma}(\tau) = \exp(-\pi\Delta\nu|\tau|)\, \exp(-j2\pi\bar{\nu}\tau)$$

$$\gamma(\tau) = e^{-\pi\Delta\nu|\tau|}$$

c. RECTANGULAR ⇒ $\hat{\mathcal{G}}(\nu) = \frac{1}{\Delta\nu}\, \mathrm{rect}\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)$

$$\underline{\gamma}(\tau) = \mathrm{sinc}(\Delta\nu\tau)\, e^{-j2\pi\bar{\nu}\tau}$$

$$\gamma(\tau) = |\mathrm{sinc}(\Delta\nu\tau)|$$



GENERAL PROPERTIES OF $\underline{\gamma}(\tau)$

- $\underline{\gamma}(\tau) = \gamma(\tau)\, e^{-j2\pi\bar{\nu}\tau}$ FOR LINE SHAPES SYMMETRIC ABOUT $\bar{\nu}$
- $\gamma(\tau)$ IS AN EVEN FUNCTION $(\gamma(\tau) = \gamma(-\tau))$

DEFINITION OF <u>COHERENCE</u> <u>TIME</u>

$$\tau_c \triangleq \int_{-\infty}^{\infty} |\underline{\gamma}(\tau)|^2\, d\tau$$ (ALWAY ROUGHY EQUAL $\frac{1}{\Delta\nu}$)

a. LOW-PRESSURE (GAUSSIAN)

$$\tau_c = \sqrt{\frac{2\ln 2}{\pi}}\, \frac{1}{\Delta\nu} = \frac{0.664}{\Delta\nu}$$

b. HIGH PRESSURE (LORENTZIAN)

$$\tau_c = \frac{1}{\pi\Delta\nu} = \frac{0.318}{\Delta\nu}$$

c. RECTANGULAR

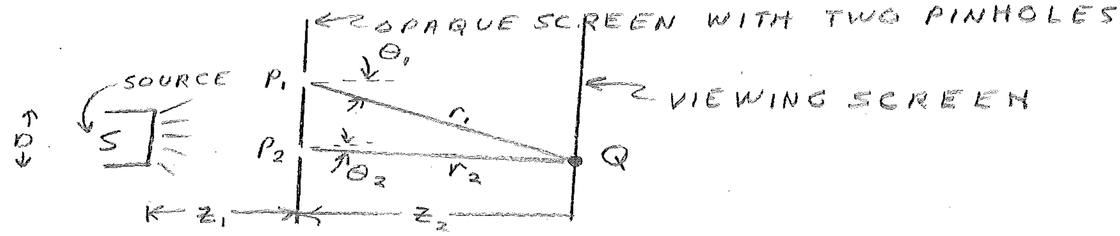
$$\tau_c = \frac{1}{\Delta\nu}$$

• 4 • FOURIER SPECTROSCOPY

AS POWER SPECTRAL DENSITY GIVES INTERFEROGRAM, SO DOES INTERFEROGRAM GIVE THE SPECTRUM. THIS IS <u>FOURIER</u> <u>SPECTROSCOPY</u>.

B. SPATIAL COHERENCE

• 1 • YOUNG'S EXPERIMENT



• 2 • MATHEMATICAL DESCRIPTION OF YOUNG'S EXPERIMENT

AT POINT Q, THE INTENSITY IS $I(Q) = \langle \underline{u}^*(Q,t)\,\underline{u}(Q,t)\rangle$

$$\underline{u}(Q,t) = \underline{K}_1\, \underline{u}(P_1, t - r_1/c) + \underline{K}_2\, \underline{u}(P_2, t - r_2/c)$$

(WE ARE HERE ASSUMING NARROWBAND LIGHT)

$$K_1 = \iint_{P_1} \chi(\theta_1)\, ds_1 / j\bar{\lambda} r \qquad K_2 = \iint_{P_2} \chi(\theta_2)\, ds_2 / j\bar{\lambda} r_2$$

NOW $I(Q) = |\underline{K}_1|^2 \langle |\underline{u}(P_1, t - \frac{r_1}{c})|^2\rangle + |\underline{K}_2|^2 \langle |\underline{u}(P_2, t - \frac{r_2}{c})|^2\rangle$

$$+ \underline{K}_1 K_2^* \langle \underline{u}(P_1, t - \tfrac{r_1}{c})\, \underline{u}(P_2, t - r_2/c)\rangle$$

$$+ K_1^* K_2 \langle u^*(P_1, t - \tfrac{r_1}{c})\, \underline{u}(P_2, t - r_2/c)\rangle$$

LET $I^{(i)}(Q) \triangleq |K_i|^2 \langle |u(P_i, t - r_i/c)|^2\rangle$, $i = 1, 2$

= LIGHT PRODUCED BY PINHOLES INDIVIDUALLY

$\Gamma_{12}(\tau) \triangleq \langle \underline{u}(P_1, t+\tau)\, \underline{u}^*(P_2, t)\rangle \Leftarrow$ <u>MUTUAL COHERENCE FUNCTION</u>

THEN $I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + \underline{K}_1 K_2^* \Gamma_{12}\left(\frac{r_1 - r_2}{c}\right) + \underline{K}_1^* K_2 \Gamma_{12}\left(\frac{r_1 - r_2}{c}\right)$

$$= I^{(1)}(Q) + I^{(2)}(Q) + 2K_1K_2 \,\mathrm{Re}\left\{\Gamma_{12}\left(\frac{r_2 - r_1}{c}\right)\right\}$$

DEFINE $\underline{\gamma}_{12}(\tau) \triangleq \Gamma_{12}(\tau) / \sqrt{\Gamma_{11}(0)\Gamma_{22}(0)} \Leftarrow$ <u>COMPLEX COHERENCE FACTOR</u>

$$|\underline{\gamma}_{12}(0)| = 1 \qquad 0 \le |\underline{\gamma}_{12}(\tau)| \le 1$$

$$I^{(1)}(Q) = K_1^2\, \Gamma_{11}(0) \qquad I^{(2)}(Q) = K_2^2\, \Gamma_{22}(0)$$

LET $\underline{\gamma}_{12}(\tau) = \gamma_{12}(\tau) \exp\{-j[2\pi\bar{\nu}\tau - \alpha_{12}(\tau)]\}$

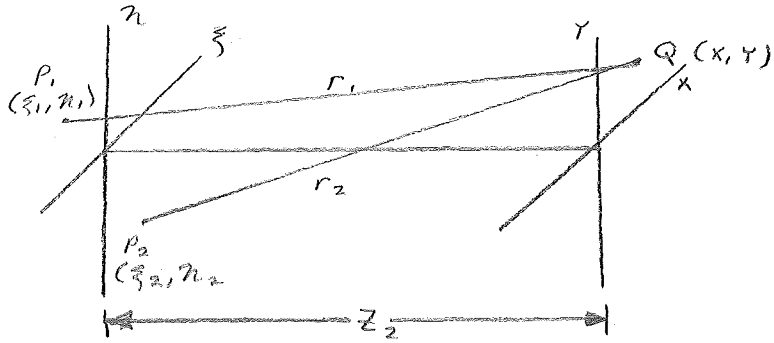
THEN $I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2\sqrt{I^{(1)}(Q) I^{(2)}(Q)}\; \gamma_{12}\left(\frac{r_2 - r_1}{c}\right)$

$$\times \cos\left(2\pi\bar{\nu}\left(\frac{r_2 - r_1}{c}\right) - \alpha\left(\frac{r_2 - r_1}{c}\right)\right)$$

$$\mathcal{V} = \frac{2\sqrt{I^{(1)}(Q) I^{(2)}(Q)}}{I^{(1)} + I^{(2)}}\; \gamma_{12}(0) \quad \text{AROUND } r_2 \simeq r_1$$

$\gamma_{12}(0)$ MEASURES THE <u>SPATIAL</u> <u>COHERENCE</u> $\left(\text{FOR } r_2 \simeq r_1\right)$

• 3 • SOME GEOMETRICAL CONSIDERATIONS



RIGOUROUSLY: $r_i = \sqrt{Z_2^2 + (\xi_i - X)^2 + (\eta_i - Y)^2}$; $i = 1, 2$

$$= Z_2 \left[1 + \left(\frac{\xi_i - X}{Z_2} \right)^2 + \left(\frac{\eta_i - Y}{Z_2} \right)^2 \right]^{1/2}$$

$$\simeq Z_2 + (\xi_i - X)^2 / 2Z_2 + (\eta_i - X)^2 / 2Z_2 ;$$

THIS APPROXIMATION IS ONLY GOOD FOR $Z_2 >> X, Y, \xi_i, \eta_i$

$$r_2 - r_1 \simeq \frac{1}{2Z_2} \left[\rho_2^2 - \rho_1^2 + 2\Delta\xi X + 2\Delta\eta Y \right]$$

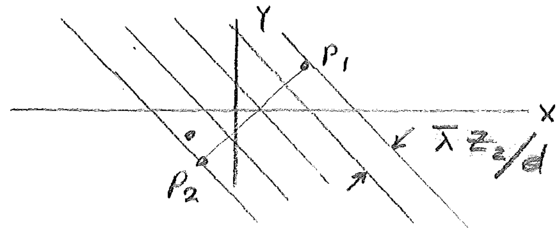
$\rho_i \triangleq \left[\xi_i^2 + \eta_i^2 \right]^{1/2}$ $\qquad \Delta\xi = \xi_1 - \xi_2 \qquad \Delta\eta = \eta_1 - \eta_2$

• FROM YOUNG'S EXPERIMENT

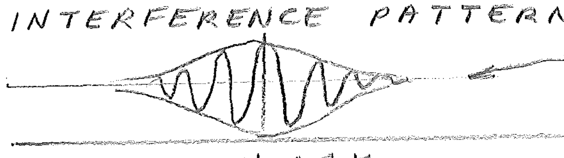
RECALL $I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2\sqrt{I^{(1)}(Q) \times I^{(2)}(Q)}\, \gamma_{12}\left(\frac{r_2 - r_1}{c}\right)$

$$\times \cos \left[2\pi \bar{\nu} \left(\frac{r_2 - r_1}{c} \right) - \alpha_{12} \left(\frac{r_2 - r_1}{c} \right) \right]$$

FROM CONSIDERATIONS ABOVE, WE SEE THAT FRINGES WILL RUN PERPENDICULAR TO P_1 & P_2 WITH PERIOD OF $\bar{\lambda} Z_2 / d \quad \ni d = \sqrt{\Delta\xi^2 + \Delta\eta^2}$



THE INTERFERENCE PATTERN $(\alpha \simeq 0,\ \gamma \sim 1)$



HALF WIDTH $= \Delta l \simeq \frac{Z_2 c}{\Delta \nu d}$

TOTAL FRINGES THERE $= N \simeq \frac{2\Delta l}{N} = \frac{2\bar{\nu}}{\Delta\nu}$

• 4 • INTERFERENCE UNDER QUASI-MONOCHROMATIC CONDITIONS

BEFORE, WE ASSUMED NARROWBAND ASSUMPTION:

$\nabla \nu \ll \bar{\nu}$. THIS ELIMINATES FRINGE WASHOUT.

WE NOW ALSO INCLUDE THE COHERENCE LENGTH ASSUMPTION:

$r_2 - r_1 \ll c\tau_c$. THIS ELIMINATES TEMPORAL COHERENCE EFFECTS

TOGETHER, THESE CONSTITUTE QUASI-MONOCHROMATIC ASSUMPTION

THIS GIVES $\underline{\Gamma}_{12}(\tau) = \underline{J}_{12}\, e^{-j2\pi\bar{\nu}\tau}$

$\underline{J}_{12} \triangleq \underline{\Gamma}_{12}(0) = \langle \underline{u}(P_1,t)\, \underline{u}^*(P_2,t)\rangle = \langle \underline{A}(P_1,t)\, \underline{A}_2^*(P_2,t)\rangle \Rightarrow$ MUTUAL INTENSI

$\underline{\mu}_{12} \triangleq \underline{\gamma}_{12}(0) = \underline{J}_{12}/\sqrt{I(P_1)I(P_2)} \Rightarrow$ COMPLEX COHERENCE FACTOR

$0 \le \mu_{12} \le 1$. J_{12} IS PHASOR AMPLITUDE OF THE SPATIAL SINUSOID FRINGE. μ_{12} IS NORMALIZED VERSION

THE YOUNG'S INTERFERENCE PATTERN BECOMES

$$I(x,y) = I^{(1)} + I^{(2)} + 2K_1K_2\, J_{12} \cos\left[\frac{2\pi}{\lambda z_2}(\Delta\xi x + \Delta\eta y) - \phi_{12}\right]$$

$$= I^{(1)} + I^{(2)} + 2\sqrt{I^{(1)}I^{(2)}}\, \mu_{12} \cos\left[\frac{2\pi}{\lambda z_2}(\Delta\xi x + \Delta\eta y) - \phi_{12}\right]$$

WHERE $J_{12} = |\underline{J}_{12}|$, $\mu_{12} = |\underline{\mu}_{12}|$ AND

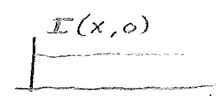
$$\phi_{12} = \arg \underline{J}_{12} - \frac{\pi}{\lambda z_2}(\rho_2^2 - \rho_1^2) = \alpha_{12} - \frac{\pi}{\lambda z_2}(\rho_2^2 - \rho_1^2)$$

NOTE THAT WE NOW HAVE CONSTANT VISIBILITY:

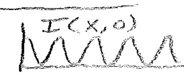
$$\mathcal{V} = \frac{2\sqrt{I^{(1)}I^{(2)}}}{I^{(1)} + I^{(2)}}\, \mu_{12}$$

$$= \mu_{12} \quad (\text{FOR } I^{(1)} = I^{(2)})$$

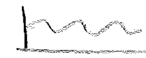
FOR $\mu_{12} = 0$ INCOHERENT



$\mu_{12} = 1$ COHERENT



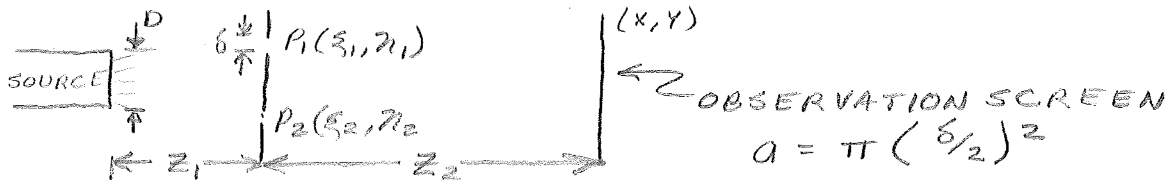
$0 < \mu_{12} < 1$ PARTIALLY COHERENT



A SHORT TABLE OF TERMS

SYMBOL	DEFINITION	NAME	MEASURES
$\underline{\Gamma}_{11}(\tau)$	$\langle \underline{u}(P_1,t+\tau)\, u^*(P_1,t)\rangle$, $\Gamma_{11}(0) = I(P_1)$	SELF COHERENCE	TEMPORAL
$\underline{\Gamma}_{12}(\tau)$	$\langle \underline{u}(P_1,t+\tau)\, u_2^*(P_1,t)$	MUTUAL COHERENCE	BOTH
$\underline{\gamma}_{12}(\tau)$	$\Gamma_{12}(\tau)/[\Gamma_{11}(0)\,\Gamma_{22}(0)]^{1/2}$	COMPLEX DEGREE OF COHERENCE	BOTH
$\underline{J}_{12}$	$\langle u(P_1,t)\, u^*(P_2,t)\rangle = \Gamma_{12}(0)$	MUTUAL INTENSITY	SPATIAL
$\underline{\mu}_{12}$	$\underline{J}_{12}/[J_{11}\, J_{22}]^{1/2}$	COMPLEX COHERENCE FACTOR	SPATIAL

• 5 • EFFECTS OF FINITE PINHOLE SIZE

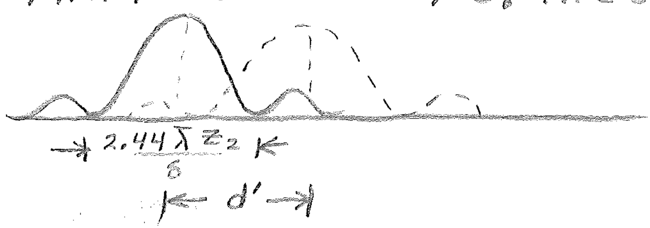


$$a = \pi\left(\delta/2\right)^2$$

THE INTENSITY DISTRIBUTION OF THE CIRCULAR PINHOLES ARE AIRY PATTERNS

$$I^{(i)}(x,y) = \left(\frac{a}{\lambda z_2}\right)^2 I(P_i)\left[\frac{2J_1\left(\frac{\pi\delta}{\lambda z_2}\sqrt{\left(x-\frac{z_1+z_2}{z_1}\xi_i\right)^2+\left(y-\frac{z_1+z_2}{z_1}\eta_i\right)^2}\right)}{\frac{\pi\delta}{\lambda z_2}\sqrt{\left(x-\frac{z_1+z_2}{z_1}\xi_i\right)^2+\left(y-\frac{z_1+z_2}{z_1}\eta_i\right)^2}}\right]^2$$

WE ARE HERE ASSUMING FRAUNHOFFER DIFFRACTION AND THAT $D \ll \lambda z_1/\delta$. THESE PATTERNS LOOK LIKE



d' = SEPARATION OF AIRY PATTERNS = $\frac{z_1+z_2}{z_1}d$

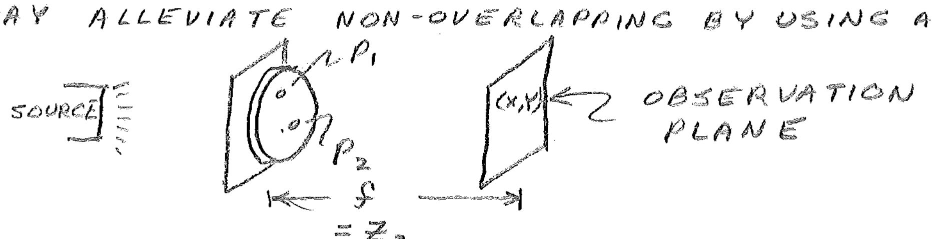
WHERE d IS THE SEPARATION BETWEEN PINHOLES

WE EXPECT NEARLY COMPLETE OVERLAP IF

$$d' \ll \frac{2.44\lambda z_2}{\delta} \quad \text{OR} \quad d \ll \frac{2.44\lambda z_1 z_2}{(z_1+z_2)\delta}$$

NOTE THAT VISIBILITY DOES NOT EQUAL μ_{12}. MUST INCLUDE A CORRECTION FACTOR.

MAY ALLEVIATE NON-OVERLAPPING BY USING A LENS



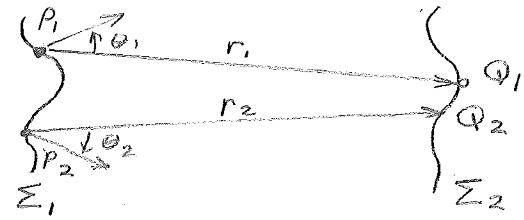
GIVES $I(x,y) = \left(\frac{a}{\lambda f}\right)^2 I\left[\frac{2J_1\left(\frac{\pi\delta}{\lambda f}\sqrt{x^2+y^2}\right)}{\frac{\pi\delta}{\lambda f}\sqrt{x^2+y^2}}\right]^2 \times 2\left[1+\mu_{12}\cos 2\pi(\Delta\xi x+\Delta\eta y)+\phi_{12}\right]$

NOTE THAT THE LENS TAKES OUT EFFECT OF ρ_2^2 & ρ_1^2

$$\phi_{12} = \arg \underline{J}_{12} = \alpha_{12}$$

C. PROPAGATION OF MUTUAL COHERENCE

1. SOLUTION BASED ON HUYGENS-FRESNEL PRINCIPLE



THE MUTUAL COHERENCE FUNCTION ON Σ_1 IS $\underline{\Gamma}(P_1, P_2; \tau)$

a. NARROWBAND LIGHT: $\Gamma(Q_1, Q_2; \tau) = \langle \underline{u}(Q_1, t+\tau)\, \underline{u}^*(Q_2, t)\rangle$

FOR NARROW BAND LIGHT, HUYGEN'S-FRESNEL PRINCIPLE IS

$$\underline{u}(Q_1, t+\tau) = \iint_{\Sigma_1} \frac{1}{j\lambda r_1}\, \underline{u}(P_1, t+\tau - \tfrac{r_1}{c})\, \chi(\theta_1)\, ds_1$$

$$\underline{u}^*(Q_2, t) = \iint_{\Sigma_1} \frac{-1}{j\lambda r_1}\, \underline{u}^*(P_2, t - \tfrac{r_2}{c})\, \chi(\theta_2)\, ds_2$$

THIS GIVES

$$\underline{\Gamma}(Q_1, Q_2, \tau) = \iint_{\Sigma_1}\iint_{\Sigma_1} \frac{\langle \underline{u}(P_1, t+\tau - \frac{r_1}{c})\, \underline{u}^*(P_2, t - \frac{r_2}{c})\rangle}{\lambda^2 r_1 r_2}\, \chi(\theta_1)\chi(\theta_2)\, ds_1 ds_2$$

$$= \iint_{\Sigma_1}\iint_{\Sigma_1} \underline{\Gamma}(P_1, P_2; \tau + \tfrac{r_2 - r_1}{c})\, \frac{\chi_1(\theta)}{\lambda r_1}\, \frac{\chi(\theta_2)}{\lambda r_2}\, ds_1 ds_2$$

b. BROADBAND LIGHT:

$$\underline{\Gamma}(Q_1, Q_2) = -\iint_{\Sigma_1}\iint_{\Sigma_1} \frac{\delta^2}{\delta\tau^2}\, \underline{\Gamma}(P_1, P_2; \tau + \tfrac{r_2 - r_1}{c})\, \frac{\chi(\theta_1)}{\lambda r_1}\, \frac{\chi(\theta_2)}{\lambda r_2}\, ds_1 ds_2$$

c. QUASI-MONOCHROMATIC LIGHT

(SAME AS NARROWBAND WITH COHERENCE LENGTH ASSUMPTION ADDED)

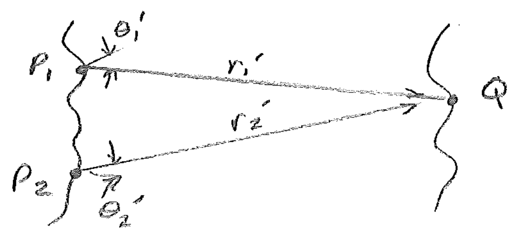
NOW, $\underline{J}(Q_1, Q_2) = \underline{\Gamma}(Q_1, Q_2; 0)$

AND $\underline{\Gamma}(P_1, P_2; \frac{r_2 - r_1}{c}) = \underline{J}(P_1, P_2)\, e^{-j\frac{2\pi}{\lambda}(r_2 - r_1)}$

GIVES: $J(Q_1, Q_2) = \iint_{\Sigma_1}\iint_{\Sigma_2} \underline{J}(P_1, P_2)\, e^{-j\frac{2\pi}{\lambda}(r_2 - r_1)}\, \frac{\chi(\theta_1)}{\lambda r_1}\, \frac{\chi(\theta_2)}{\lambda r_2}\, ds_1 ds_2$

THE INTENSITY DISTRIBUTION ON Σ_2 IS FOUND BY LETTING $Q_1 \to Q_2 = Q$:

$$I(Q) = \iint_{\Sigma_1}\iint_{\Sigma_1} \underline{J}(P_1, P_2)\, e^{-j\frac{2\pi}{\lambda}(r_2' - r_1')}\, \frac{\chi(\theta_1)}{\lambda r_1'}\, \frac{\chi(\theta_2)}{\lambda r_2'}\, ds_1 ds_2$$



ALL OF THE ABOVE ARE BASED ON THE HUYGEN'S FRESNEL PRINCIPLE, AND THUS, CORRESPONDING ASSUMPTIONS MUST FOLLOW.

• 2 • WAVE EQUATION GOVERNING PROPAGATION OF MUTUAL COHERENCE

IN FREE SPACE, THE WAVE EQUATION IS

$$\nabla^2 U^r(P,t) = \frac{1}{c^2}\frac{\delta^2}{\delta t^2} U^r(P,t)$$

$$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \Leftarrow \underline{\text{LAPLACIAN}}\ \underline{\text{OPERATOR}}$$

$$U^i = \mathcal{H}[U^r] = \underline{\text{HILBERT}}\ \underline{\text{XFORM}}[U^r] = \mathrm{Im}\,\underline{U}$$

$$\Rightarrow \nabla^2 U^i(P,t) = \frac{1}{c^2}\frac{\delta^2}{\delta t^2} U^i(P,t)$$

COMBINING: $\nabla^2 \underline{U}(P,t) = \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\underline{U}(P,t)$

NOW $\underline{\Gamma}_{12}(\tau) = \langle \underline{U}_1(t+\tau)\,\underline{U}_2^*(t)\rangle$

LET $\nabla_1^2 \triangleq \frac{\delta^2}{\delta x_1^2} + \frac{\delta^2}{\delta y_1^2} + \frac{\delta^2}{\delta z_1^2} \Leftarrow$ OPERATES ON P_1

THEN $\nabla_1^2 \underline{\Gamma}_{12}(\tau) = \nabla_1^2 \langle U_1(t+\tau)\,U_2^*(t)\rangle$

BUT $\nabla^2 \underline{U}_1(t+\tau) = \frac{1}{c^2}\,\delta^2 \underline{U}_1(t+\tau)/\delta(t+\tau)^2$

$$\Rightarrow \nabla_1^2 \underline{\Gamma}_{12}(\tau) = \frac{1}{c^2}\frac{\delta^2}{\delta(t+\tau)^2}\langle U_1(t+\tau)\,U_2(t)\rangle$$

$$= \frac{1}{c^2}\frac{\delta^2}{\delta(t+\tau)^2}\Gamma_{12}(\tau)$$

THUS $\nabla_1^2 \underline{\Gamma}_{12}(\tau) = \frac{1}{c^2}\frac{\delta^2}{\delta\tau^2}\underline{\Gamma}_{12}(\tau)$

ALSO $\nabla_2^2 \underline{\Gamma}_{12}(\tau) = \frac{1}{c^2}\frac{\delta^2}{\delta\tau^2}\underline{\Gamma}_{12}(\tau)$

UNDER THE QUASIMONOCHROMATIC ASSUMPTION, THESE RELATIONSHIPS BECOME $\underline{\text{HELMHOLTZ}}$ $\underline{\text{EQUATIONS}}$

$$\nabla_1^2 \underline{J}_{12} + \bar{k}^2 \underline{J}_{12} = 0 \quad , \quad \nabla_2^2 \underline{J}_{12} + \bar{k}^2 \underline{J}_{12} = 0$$

D. LIMITING FORMS OF THE MUTUAL COHERENCE FUNCTION

• 1• A COHERENT FIELD

$|\gamma_{12}(\tau)| = 1 \ \forall\ P_1, P_2, \tau$. ALLOWS ONLY PERFECT MONOCHRO. LIGHT

MORE USEFUL DEFINITION IS: $\max_{\tau} |\underline{\gamma}_{12}(\tau)| = 1 \ \forall\ P_1, P_2$

$$|\gamma_{12}(\tau_{12})| = \frac{|\langle U(P_1, t+\tau_{12})\, U^*(P_2, t)\rangle|}{[\langle |U(P_1, t+\tau_{12})|^2\rangle \langle |U(P_2, t)|^2\rangle]^{1/2}}$$

$$= \frac{|\langle A(P_1, t+\tau_{12})\, A^*(P_2, t)\rangle|}{[\langle |A(P_1, t+\tau_{12})|^2\rangle \langle |\underline{A}(P_2, t)|^2\rangle]^{\frac{1}{2}}}$$

$$\left|\int f(t) g^*(t)\, dt\right| \le \left[\int |f(t)|^2 dt \int |g(t)|^2 dt\right]^{\frac{1}{2}}$$

THIS IS SCHWARZ'S INEQUALITY. WE HAVE EQUALITY ONLY IF $g(t) = k f(t)$

THUS $|\gamma_{12}(\tau_{12})| = 1$ ONLY IF $\underline{A}(P_2, t) = \underline{k}_{12} A(P_1, t+\tau_{12})$

WAVE IS COHERENT IF $\forall\ P_1, P_2\ \exists\ \tau_{12} \ni |\gamma_{12}(\tau_{12})| = 1$

UNDER QUASIMONOCHROMATIC CONDITIONS, $\forall\ P_1, P_2$, THE SAME τ_{12} IS REQUIRED $\Rightarrow \underline{A}(P_2, t) = \underline{k}_{12}\, \underline{A}(P_1, t)$

• FOR FULLY-COHERENT QUASIMONOCHROMATIC LIGHT

LET P_0 BE A REFERENCE POINT, THEN

$$\underline{A}(P_1, t) = \underline{A}(P_1) \frac{\underline{A}(P_0, t)}{[I(P_0)]^{\frac{1}{2}}}\ ; \ \underline{A}(P_2, t) = A(P_2) \frac{\underline{A}(P_0, t)}{[I(P_0)]^{1/2}}$$

THEN $\underline{J}_{12} = \langle \underline{A}(P_1, t)\, \underline{A}^*(P_2, t)\rangle = \underline{A}(P_1)\, \underline{A}^*(P_2)$

$$\underline{\mu}_{12} = \exp j\,[\phi(P_1) - \phi(P_2)]$$

WHERE $\phi(P_i) = \arg[\underline{A}(P_i)]$; $i = 1, 2$

AND THE RESULTING FRINGE PATTERN IS

$$I(Q) = I^{(1)} + I^{(2)} + 2\sqrt{I^{(1)} I^{(2)}} \cos\left[\frac{2\pi}{\lambda z}(\Delta\xi X + \Delta\eta Y) - \phi_{12}\right]$$

(VALID ONLY FOR ITTY-BITTY PINHOLES)

NOTE, FOR $I^{(1)} = I^{(2)}$, $\mathcal{V} = 1$

• 2• AN INCOHERENT FIELD

$|\gamma_{12}(\tau)| = 0 \ \forall\ P_1 \ne P_2 \ \forall\ \tau$

BUT THIS GIVES $\underline{\Gamma}(Q_1, Q_2; \tau) = 0 \Rightarrow$ NO LIGHT PROPAGATION

WE HAVE FAILED TO TAKE INTO ACCOUNT EVANESCENT WAVES

DOING SO TURNS OUT TO GIVE

$$J(P_1, P_2) = \sqrt{I(P_1) I(P_2)} \left[\frac{2 J_1\left(\bar{k}\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}\right)}{\bar{k}\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}}\right]$$

$$\simeq \kappa I(P_1)\, \delta(X_1 - X_2, Y_1 - Y_2) \quad ; \ \kappa = \lambda^2/\pi$$

E. THE VAN CITTERT-ZERNIKE THEOREM

1. MATHEMATICAL DERIVATION

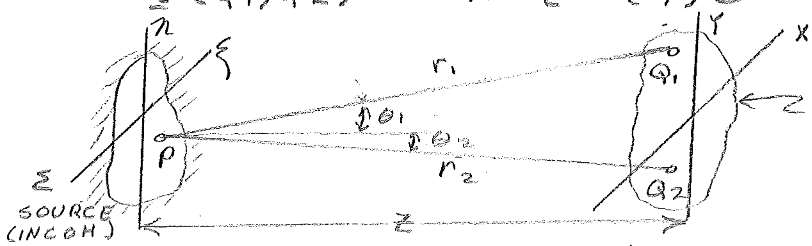
RESTRICT ATTENTION TO QUASIMONOCHROMATIC LIGHT

$$\underline{J}(Q_1,Q_2)=\iint_\Sigma\iint_\Sigma \underline{J}(P_1,P_2)\,e^{-j\frac{2\pi}{\lambda}(r_2-r_1)}\,\frac{\chi(\theta_1)}{\lambda r_1}\,\frac{\chi(\theta_2)}{\lambda r_2}\,ds_1\,ds_2$$

FURTHER RESTRICT ATTENTION TO INCOHEREN LIGHT

$$\underline{J}(P_1,P_2)=K\,I(P_1)\,\delta(|P_1-P_2|)$$

$$\Rightarrow \underline{J}(Q_1,Q_2)=K/\lambda^2\iint_\Sigma I(P_1)\,e^{-j\frac{2\pi(r_2-r_1)}{\lambda}}\,\frac{\chi(\theta_1)}{r_1}\frac{\chi(\theta_2)}{r_2}\,ds$$



APPROXIMATIONS: $\frac{1}{r_1}\frac{1}{r_2}\doteq\frac{1}{Z^2}$; $\chi(\theta_1)=\chi(\theta_2)=1$

$$\Rightarrow \underline{J}(Q_1,Q_2)=\frac{K}{(\lambda Z)^2}\iint_\Sigma I(P_1)\,e^{-j\frac{2\pi}{\lambda}(r_2-r_1)}\,ds$$

PARAXIAL APPROX: $r_i = Z+\frac{(X_i-\xi)^2+(Y_i-\eta)^2}{2Z}$; $i=1,2$

ALSO, LET $\Delta X = X_1-X_2$, $\Delta Y = Y_1-Y_2$

$$\Rightarrow \underline{J}(X_1,Y_1;X_2,Y_2)=\frac{K\,e^{j\psi}}{(\lambda Z)^2}\int_{-\infty}^{\infty}\!\!\int e^{j\frac{2\pi}{\lambda Z}(\Delta X\xi+\Delta Y\eta)}\,I(\xi,\eta)\,d\xi\,d\eta$$

$$\psi=\frac{\pi}{\lambda Z}\left[(X_2^2+Y_2^2)-(X_1^2+Y_1^2)\right]=\frac{\pi}{\lambda Z}\left[\rho_2^2-\rho_1^2\right]$$

EQUIVALENTLY, WE MAY EXPRESS VAN-CITTERT ZERNIKE AS

$$\underline{\mu}(X_1,Y_1;X_2,Y_2)=\frac{e^{-j\psi}\int_{-\infty}^{\infty}\!\!\int I(\xi,\eta)\,e^{j\frac{2\pi}{\lambda Z}(\Delta X\xi+\Delta Y\eta)}\,d\xi\,d\eta}{\int_{-\infty}^{\infty}\!\!\int I(\xi,\eta)\,d\xi\,d\eta}$$

IN MOST CASES, $I(X_1,Y_1)=I(X_2,Y_2)$ AND $\underline{\mu}$ REPRESENTS THE CLASSICAL FRINGE VISIBILITY

2. DISCUSSION

a. THE MUTUAL INTENSITY IS FOURIER XFORM OF SOURCE INTENSITY

b. ANALAGOUS TO τ_c, WE DEFINE THE <u>COHERENCE</u> <u>AREA</u> AS

$$A_C\triangleq\int_{-\infty}^{\infty}\!\!\int|\underline{\mu}(\Delta X,\Delta Y)|^2\,d\Delta X\,d\Delta Y$$

FOR UNIFORMLY BRIGHT INCOHERENT SOURCE, $A_C=\frac{(\bar{\lambda}Z)^2}{A_S}$

c. WE CAN IGNORE THE $e^{-j\psi}=e^{-j\frac{\pi}{\lambda Z}(\rho_2^2-\rho_1^2)}$ TERM WHEN

$\Rightarrow Z \gg 2(\rho_2^2-\rho_1^2)/\bar{\lambda} \Rightarrow \psi \ll \frac{\pi}{2} \Rightarrow e^{-j\psi}\simeq 1$

$\Rightarrow Q_1$ & Q_2 ARE MAINTAINED @ EQUAL DISTANCE FROM OPTICAL AXIS $\Rightarrow \psi \doteq 0$

3 AN EXAMPLE

LET SOURCE BE UNIFORMLY ILLUMINATED CIRCLE:

$$I(\xi,\eta) = I_0 \operatorname{circ} \frac{\sqrt{\xi^2+\eta^2}}{a} \Leftarrow \text{RADIUS } a$$

$$\ni \operatorname{circ} w = \begin{cases} 1 & ; w<1 \\ \frac{1}{2} & ; w=1 \\ 0 & ; w>1 \end{cases}$$

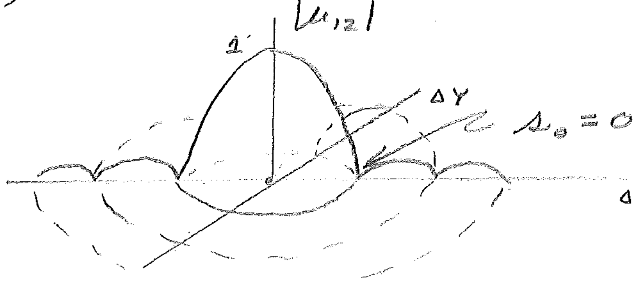
$$\mathcal{F}\left[\operatorname{circ}\frac{\sqrt{\xi^2+\eta^2}}{a}\right] = a^2 \frac{J_1(2\pi a\sqrt{V_x^2+V_y^2})}{a\sqrt{V_x^2+V_y^2}}$$

SUBSTITUTING $V_X = \frac{\Delta X}{\lambda z}$ $V_Y = \frac{\Delta Y}{\lambda z}$, WE GET

$$\underline{J}(X_1,Y_1;X_2,Y_2) = \frac{\pi a^2 I_0 K\, e^{-j\psi}}{(\lambda z)^2}\left[\frac{2\, J_1\left(\frac{2\pi a}{\lambda z}\sqrt{\Delta X^2+\Delta Y^2}\right)}{\frac{2\pi a}{\lambda z}\sqrt{\Delta X^2+\Delta Y^2}}\right]$$

OR

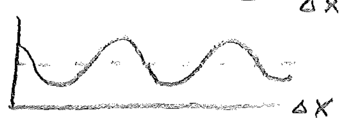
$$\mu(X_1,Y_1;X_2,Y_2) = e^{-j\psi}\left[\frac{2\, J_1\left(\frac{2\pi a}{\lambda z}\sqrt{\Delta X^2+\Delta Y^2}\right)}{\frac{2\pi a}{\lambda z}\sqrt{\Delta X^2+\Delta Y^2}}\right]$$



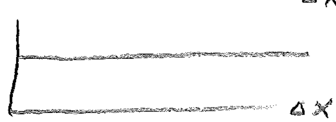
$\Delta_0 = 0.610 \frac{\lambda z}{a}$

IF Q_1 & Q_2 ARE EQUADISTANT FROM OPTICAL AXIS & $\Delta Y = 0$

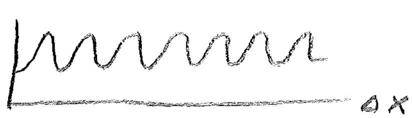
$0 < \Delta X << 0.610\, \lambda z / a = \Delta_0$



$0 < \Delta X < \Delta_0$



$\Delta X = \Delta_0$



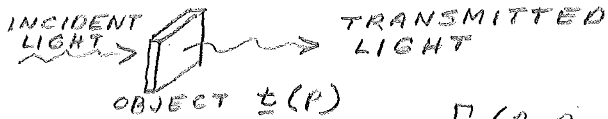
ΔX A BIT BIGGER THAN Δ_0

(NOTE PHASE REVERSAL)

III. PARTIAL COHERENCE EFFECTS IN IMAGE FORMATION

A. SOME PRELIMINARY CONSIDERATIONS

1. EFFECTS OF A TRANSMITTING OBJECT ON MUTUAL COHERENCE

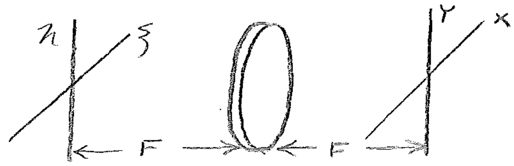


$$\Rightarrow \Gamma_{TRANS}(P_1,P_2;\tau) = t(P_1)\,t^*(P_2)\,\Gamma_{INC}(P_1,P_2;\tau)$$

OR, FOR QUASIMONOCHROMATIC LIGHT

$$J_{TRANS}(P_1,P_2) = t(P_1)\,t^*(P_2)\,J_{INC}(P_1,P_2)$$

2. FOCAL PLANE - FOCAL PLANE COHERENCE RELATIONS



FOR MONOCHROMATIC LIGHT, THE PHASER AMPLITUDES ARE

$$A_F(x,y) = \frac{1}{j\lambda F}\int_{-\infty}^{\infty}\!\!\int A_0(\xi,\eta)\, e^{-j\frac{2\pi}{\lambda F}(x\xi+y\eta)}\,d\xi\, d\eta$$

WHEN LIGHT IS QUASI-MONO, WE REPLACE $A_0(\xi,\eta)$ BY $A_0(\xi,\eta;t)$:

$$A_F(x,y;t) = \frac{1}{j\lambda F}\int_{-\infty}^{\infty}\!\!\int A_0(\xi,\eta;t-\tau_0)\, e^{-j\frac{2\pi}{\lambda F}(x\xi+y\eta)}\,d\xi\, d\eta$$

$\Rightarrow \tau_0$ = PROPAGATION TIME $\simeq 2F/c$

THE RESULTING MUTUAL INTENSITY IS

$$J_F(x_1,y_1;x_2,y_2) = \langle A_F(x_1;y_1;t)\, A_F^*(x_2,y_2;t)\rangle$$

$$= \frac{1}{(\lambda F)^2}\int_{-\infty}^{\infty}\!\!\int\int_{-\infty}^{\infty}\!\!\int \langle A_0(\xi_1,\eta_1;t-\tau_{01})\, A_0^*(\xi_2,\eta_2;t-\tau_{02})\rangle$$

$$\times \exp\left\{-j\frac{2\pi}{\lambda F}(x_1\xi_1+y_1\eta_1-x_2\xi_2-y_2\eta_2)\right\}d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2$$

UNDER QUASIMONO CONDITIONS, $\tau_{01}-\tau_{02} \ll 1/\Delta\nu$ AND

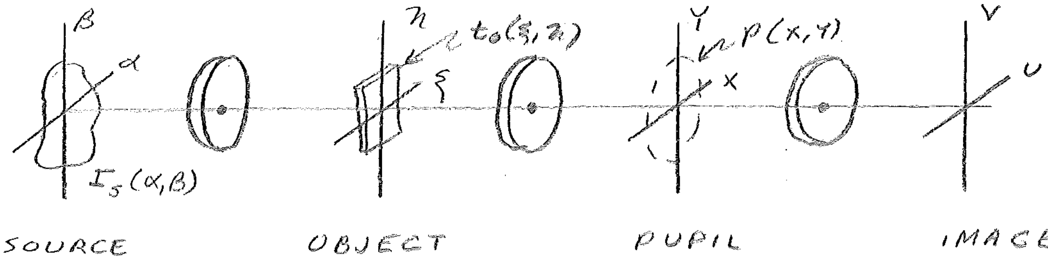
$$\langle A_0(\xi_1,\eta_1;t-\tau_{01})\, A^*(\xi_2,\eta_2;t-\tau_{02})\rangle = J_0(\xi_1,\eta_1;\xi_2,\eta_2)$$

THUS, THE MUTUAL INTENSITIES ARE RELATED BY A FOUR-DIMENSIONAL FOURIER XFORM:

$$J_F(x_1,y_1;x_2,y_2) = \frac{1}{(\lambda F)^2}\int_{-\infty}^{\infty}\!\!\int\int_{-\infty}^{\infty}\!\!\int J_0(\xi_1,\eta_1;\xi_2,\eta_2)$$

$$\times \exp\left\{-j\frac{2\pi}{\lambda F}(x_1\xi_1+y_1\eta_1-x_2\xi_2-y_2\eta_2)\right\}d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2$$

B. IMAGE CALCULATION BY INTEGRATION OVER THE SOURCE



HOPKIN'S METHOD:

DIVIDE SOURCE INTENSITY INTO $d\sigma = d\alpha d\beta$

IF $\underline{A}_s(\alpha,\beta)$ REPRESENTS COMPLEX APLITUDE OF SOURCE;

$$A_0(\xi,\eta;\alpha,\beta;t) = t_0(\xi,\eta)\, e^{-j2\pi(\xi\alpha+\eta\beta)} \cdot \frac{1}{\lambda F}\, \underline{A}_s(\alpha,\beta,t-\tau)\, d\alpha d\beta$$

$$\langle |A_s(\alpha,\beta;t)|^2 \rangle = I_s(\alpha,\beta)$$

LET $\underline{K}$ REPRESENT AMPLITUDE-SPREAD FUNCTION FROM OBJECT TO IMAGE PLANE. THEN

$$\underline{A}_i(u,v;\alpha,\beta;t) = \int_{-\infty}^{\infty}\!\!\int \underline{K}(u-\xi, v-\eta)\, \underline{A}_0(\xi,\eta;\alpha,\beta;t-\tau_2)\, d\xi d\eta$$

$$\Rightarrow I_i(u,v;\alpha,\beta) = \langle |\underline{A}_i(u,v;\alpha,\beta;t)|^2 \rangle$$

$$= \int\int_{-\infty}^{\infty}\!\!\int\int \underline{K}(u-\xi_1; v-\eta_1)\, \underline{K}^*(u-\xi_2; v-\eta_2)$$

$$\times \langle \underline{A}_0(\xi_1,\eta_1;\alpha,\beta;t-\tau_2)\, \underline{A}_0^*(\xi_2,\eta_2;\alpha,\beta;t-\tau_2') \rangle d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

UNDER QUASI-MONO, THE TIME AVERAGE IS MUTUAL INTENSITY

$$J_0(\xi_1,\eta_1;\xi_2,\eta_2;\alpha,\beta) = t_0(\xi_1,\eta_1)\, t_0^*(\xi_2,\eta_2)\, \left(\frac{1}{\lambda F}\right)^2$$

$$\times \exp\left\{ -\frac{j2\pi}{\lambda F}(\xi_1\alpha + \eta_1\beta - \xi_2\alpha - \eta_2\beta) \right\} I_s(\alpha,\beta)\, d\alpha d\beta$$

$$\Rightarrow \underline{I}_i(u,v;\alpha,\beta) = \int\int_{-\infty}^{\infty}\!\!\int\int \underline{K}(u-\xi_1, v-\eta_1)\, \underline{K}^*(u-\xi_2, v-\eta_2)\, t_0(\xi_1,\eta_1)\, t_0^*(\xi_2,\eta_2)$$

$$\times \left[\frac{1}{(\lambda F)^2}\, e^{-j2\pi/\lambda F(\alpha\Delta\xi + \beta\Delta\eta)}\, I_s(\alpha,\beta)\, d\alpha d\beta \right] d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

INTEGRATING OVER THE SOURCE (ALL α & β):

$$I_i(u,v) = \int_{-\infty}^{\infty}\!\!\int I_i(u,v;\alpha,\beta)\, d\alpha d\beta$$

$$= \int\int_{-\infty}^{\infty}\!\!\int\int \underline{K}(u-\xi_1, v-\eta_1)\, K^*(u-\xi_2, v-\eta_2)\, t_0(\xi_1,\eta_1)\, t_0^*(\xi_2,\eta_2)$$

$$\left[\frac{1}{(\lambda F)^2} \int_{-\infty}^{\infty}\!\!\int I_s(\alpha,\beta)\, e^{-j\frac{2\pi}{\lambda F}(\alpha\Delta\xi + \beta\Delta\eta)}\, d\alpha d\beta \right] d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

BUT, FROM VAN CITTERT-ZERNIKE THEOREM, THE TERM INSIDE THE BRACKETS IN MUTUAL INTENSITY OF THE LIGHT ON THE OBJECT PLANE

$$\Rightarrow I_i(u,v) = \int\int_{-\infty}^{\infty}\!\!\int\int \underline{K}(u-\xi_1, v-\eta_1)\, K^*(u-\xi_2, v-\eta_2)$$

$$t_0(\xi_1,\eta_1)\, t_0^*(\xi_2,\eta_2)\, \underline{J}_0(\Delta\xi, \Delta\eta)\, d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

C. ALTERNATIVE METHOD FOR IMAGE CALCULATION

1. DERIVATION

$$A_i(u,v;t) = \iint_{-\infty}^{\infty} \underline{K}(u-\xi; v-\eta)\,\underline{A}_0(\xi,\eta;t-\tau)\,d\xi\,d\eta$$

NOTE: THIS APPROACH ALLOWS FOR ALL OF THE LIGHT.

NOW $I_i(u,v) = \langle |\underline{A}(u,v;t)|^2 \rangle$

$$= \iiiint_{-\infty}^{\infty} \underline{K}(u-\xi_1, v-\eta_1)\,\underline{K}^*(u-\xi_2, v-\eta_2) \times \langle \underline{A}_0(\xi_1,\eta_1;t-\tau_1)\,\underline{A}_0^*(\xi_2,\eta_2;t-\tau_2)\rangle d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2$$

FOR QUASI-MONO, $\langle \cdot \rangle$ IS MUTUAL INTENSITY OF LIGHT TRANSMITTED BY THE OBJECT:

$$I_i(u,v) = \iiiint_{-\infty}^{\infty} \underline{K}(u-\xi_1, v-\eta_1)\,\underline{K}^*(u-\xi_2, v-\eta_2)\,\underline{t}_0(\xi,\eta)\,\underline{t}_0^*(\xi_2,\eta_2) \times \underline{J}_0(\xi_1,\eta_1;\xi_2,\eta_2)\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2$$

NOTE: INCOHENT SOURCE HAS NOT BEEN ASSUMED. FOR INCOHERENT SOURCE, $\underline{J}_0 = \underline{J}_0(\Delta\xi, \Delta\eta)$

2. COHERENT AND INCOHERENT LIMITS

a. COHERENT: SOURCE IS A SINGLE POINT

$$\Rightarrow \underline{J}_0(\xi_1,\eta_1;\xi_2,\eta_2) = \underline{A}(\xi,\eta)\,\underline{A}^*(\xi_2,\eta_2)$$

$$\underline{A}_0(\xi,\eta) = \underline{A}(\xi,\eta)\,t_0(\xi,\eta)$$

$$\Rightarrow I_i(u,v) = \left| \iint_{-\infty}^{\infty} \underline{K}(u-\xi, v-\eta)\,\underline{A}_0(\xi,\eta)\,d\xi\,d\eta \right|^2$$

NOTE: THE SYSTEM IS LINEAR IN AMPLITUDE

b. INCOHERENT: $\underline{J}_0(\xi_1,\eta_1;\xi_2,\eta_2) = K\,I(\xi_1,\eta_1)\,\delta(\xi_1-\xi_2, \eta_1-\eta_2)$

$$\Rightarrow I_i(u,v) = K \iint_{-\infty}^{\infty} |\underline{K}(u-\xi, v-\eta)|^2\, I_0(\xi,\eta)\,d\xi\,d\eta$$

WHERE $I_0(\xi,\eta) = I(\xi,\eta)\,|t_0(\xi,\eta)|^2$

NOTE: SYSTEM IS LINEAR IN INTENSITY

ONE MAY SHOW THE RELATIONSHIP BETWEEN MUTUAL INTENSITIES IS

$$J_i(u_1,v_1;u_2,v_2) = \iiiint_{-\infty}^{\infty} \underline{K}(u_1-\xi_1, v_1-\eta_1)\,K^*(u_2-\xi_2; v_2-\eta_2) \times t_0(\xi_1,\eta_1)\,t_0^*(\xi_2,\eta_2)\,J_0(\xi_1,\eta_1;\xi_2,\eta_2)\,d\xi_1\,d\xi_2\,d\eta_1\,d\eta_2$$

AGAIN, NOTE LINEARITY OF MUTUAL INTENSITY RELATIONSHIP FOR INCOHERENT ILLUMINATION.

• 3 • AN EXAMPLE - THE IMAGE OF TWO CLOSELY SPACED POINTS

LET $t_o(\xi,\eta) = a\,\delta(\xi - \frac{X}{2};\eta) + a\,\delta(\xi + \frac{X}{2};\eta)$

WHERE X IS THE SEPARATION TWIXT PINHOLES

$$\Rightarrow I_i(u,v) = I\Big[|\underline{K}(u-\tfrac{X}{2},v)|^2 + |\underline{K}(u+\tfrac{X}{2};v)|^2$$
$$+ 2\,\mathrm{Re}\,[\underline{\mu}\,\underline{K}(u-\tfrac{X}{2};v)\,\underline{K}^*(u+\tfrac{X}{2};v)]\Big]$$

WHERE $I = a^2 \underline{J}_o(0,0)$; $\underline{\mu} = \frac{1}{I} a^2 \underline{J}_o(X,0)$

AND WE HAVE USED THE RELATION $\underline{J}_o(-X,0) = \underline{J}_o^*(X,0)$

NOW $\underline{K}(u,v) = \frac{1}{\lambda F} \iint_{-\infty}^{\infty} \underline{P}(x,y)\, e^{-j\frac{2\pi}{\lambda F}(ux+vy)}\,dx\,dy$

LET $\underline{P}(x,y)$ BE (HERMITIAN) CIRCULAR PUPIL OF RADIUS r_p

$$\Rightarrow \underline{K}(u,v) = K(\rho) = \frac{k r_p^2}{2F}\left[2\,\frac{J_1(k r_p \rho/F)}{k r_p \rho/F}\right]$$

$$\Rightarrow I_i(u,v) = I\left[K^2(u-\tfrac{X}{2},v) + K^2(u+\tfrac{X}{2},v)\right]$$
$$+ 2\mu\, K(u-\tfrac{X}{2},v)\,K(u+\tfrac{X}{2},v)\cos\phi$$

$\Rightarrow \mu = |\underline{\mu}|$ & $\phi = \arg \underline{\mu}$

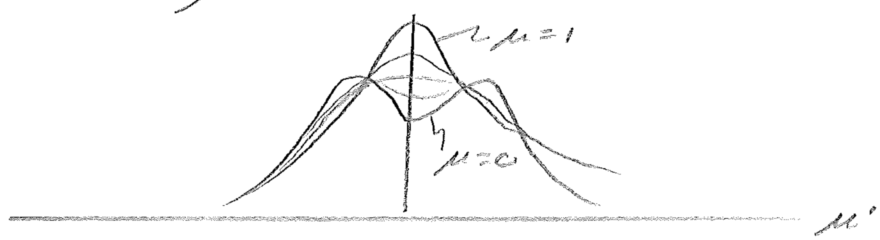
IF WE ASSUME

(1) NO ABERRATIONS

(2) $\underline{\mu}$ IS REAL AND POSITIVE ($\underline{\mu} = \mu$)

(3) SEPARATION IS $X = \frac{2F\lambda}{\pi r_p}$ ≃ <u>RAYLEIGH DISTANCE</u>

LET $\mu' = 2\pi r_p \mu / \lambda F$



D. YET ANOTHER APPROACH TO THE PROBLEM

1. MUTUAL INTENSITY IN THE IMAGE PLANE

CONSIDER MUTUAL INTENSITY INCIDENT ON PUPIL PLANE

(ADOPT NOTATION J' = TRANSMITTED & J = INCIDENT)

$$\underline{J}_p(x_1,y_1;x_2,y_2) = \frac{1}{(\lambda F)^2}\iiiint_{-\infty}^{\infty} J_o'(\xi_1,\eta_1;\xi_2,\eta_2)$$
$$\times \exp\left[-j\frac{2\pi}{\lambda F}(x_1\xi_1 + y_1\eta_1 - x_2\xi_2 - y_2\eta_2)\right]d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2$$

$$J_o'(\xi_1,\eta_1;\xi_2,\eta_2) = t_o(\xi_1,\eta_1)t_o^*(\xi_2,\eta_2)\,\underline{J}_o(\Delta\xi,\Delta\eta)$$

$$\Rightarrow \underline{J}_p(x_1,y_1;x_2,y_2) = \frac{1}{(\lambda F)^2}\iiiint_{-\infty}^{\infty} t_o(\xi_1,\eta_1)t_o^*(\xi_2,\eta_2)\,J_o(\Delta\xi,\Delta\eta)$$
$$\exp\left[-j\frac{2\pi}{\lambda F}(x_1\xi_1 + y_1\eta_1 - x_2\xi_2 - y_2\eta_2)\right]d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2$$

LET $\xi_2 = \xi_1 - \Delta\xi \qquad \eta_2 = \eta_1 - \Delta\eta$

$$\Rightarrow \underline{J}_p(x_1,y_1;x_2,y_2) = \frac{1}{(\lambda F)^2}\iiiint_{-\infty}^{\infty} \underline{J}_o(\Delta\xi,\Delta\eta)\,\underline{t}_o(\xi_1,\eta_1)t_o^*(\xi_1-\Delta\xi,\eta_1-\Delta\eta)$$
$$\times \exp\left[-j\frac{2\pi}{\lambda F}(x_1\xi_1 + y_1\eta_1 - x_2\xi_1 + x_2\Delta\xi - y_1\eta_1 + y_2\Delta\eta)\right]d\xi_1\,d\eta_1\,d\Delta\xi\,d\Delta\eta$$

DEFINE $K_o = \iint_{-\infty}^{\infty} |\underline{t}_o(\xi,\eta)|^2\,d\xi\,d\eta$

GIVES

$$\underline{J}_p(x_1,y_1;x_2,y_2) = \frac{K_o}{(\lambda F)^2}\iint_{-\infty}^{\infty} J_o(\Delta\xi,\Delta\eta)\,T(\Delta\xi,\Delta\eta;\Delta x,\Delta y)$$
$$\times \exp\left[-j\frac{2\pi}{\lambda F}(x_2\Delta\xi + y_2\Delta\eta)\right]d\Delta\xi\,d\Delta\eta$$

WHERE THE <u>AMBIGUITY FUNCTION</u> IS

$$T(\Delta\xi,\Delta\eta;\Delta x,\Delta y) \triangleq \frac{1}{K_o}\iint_{-\infty}^{\infty} \underline{t}_o(\xi_1,\eta_1)t_o^*(\xi_1-\Delta\xi,\eta_1-\Delta\eta)$$
$$\times e^{-j\frac{2\pi}{\lambda F}(\Delta x\,\xi_1 + \Delta y\,\eta_1)}\,d\xi_1\,d\eta_1$$

THE MUTUAL INTENSITY TRANSMITTED BY THE PUPIL IS

$$J_p'(x_1,y_1;x_2,y_2) = \underline{P}(x_1,y_1)P^*(x_2,y_2)\,\underline{J}_p(x_1,y_1;x_2,y_2)$$

2. THE IMAGE INTENSITY AND ITS FOURIER SPECTRUM

WE WISH TO FIND $I_i(u,v)$ FROM $\underline{J}_p'$.

$$I(u,v) = \underline{J}_i(u,v;\,u;\,v)$$
$$= \left(\frac{1}{\lambda F}\right) \iiiint_{-\infty}^{\infty} J_p'(x_1, y_1; x_2, y_2)\, e^{-\frac{j2\pi}{\lambda F}(ux_1 + vy_1 - ux_2 - vy_2)}\, dx_1 dx_2 dy_1 dy_2$$

OR, EQUIVALENTLY, FOR $x_2 = x_1 - \Delta x$ AND $y_2 = y_1 - \Delta y$:

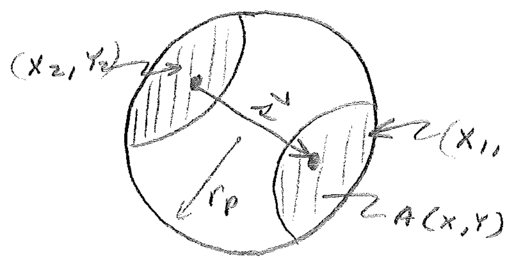
$$I_i(u,v) = \left(\frac{1}{\lambda F}\right)^2 \iint_{-\infty}^{\infty} \left[\iint_{-\infty}^{\infty} \underline{J}_p'(x_1, y_1; x_1 - \Delta x, y_1 - \Delta y)\, dx_1 dy_1\right]$$
$$\times \exp\left[-\frac{j2\pi}{\lambda F}(u\Delta x + v\Delta y)\right] d\Delta x\, d\Delta y$$

IF $\underline{\mathcal{D}}(V_x, V_y)$ = FOURIER SPECTRUM OF $I(u,v)$ WHERE $V_x = \Delta x / \lambda F$ & $V_y = \Delta y / \bar{\lambda} F$, IT FOLLOWS THAT

$$\underline{\mathcal{D}}(V_x, V_y) = \iint_{-\infty}^{\infty} J_p'(x_1, y_1; x_1 - \bar{\lambda} F V_x, y_1 - \bar{\lambda} F V_y)\, dx_1 dy_1$$

V_x & V_y DENONE SPATIAL FREQUENCY OF RESULTING FRINGE

THE ACT OF INTEGRATING OVER x_1 & y_1 MAY BE VIEWED AS A SLIDING VECTOR $\vec{\lambda} = (\Delta x, \Delta y)$ SLIDING ACROSS THE PUPIL IN ALL POSSIBLE WAYS



(V_x, V_y) PAIR ARE GIVEN BY $\vec{\lambda}$

TAKING INTO ACCOUNT THE FINITE PUPIL:

$$\mathcal{D}(V_x, V_y) = \iint_{-\infty}^{\infty} P(x_1, y_1) P^*(x_1 - \bar{\lambda} F V_x, y_1 - \bar{\lambda} F V_y)$$
$$\cdot\, J_p(x_1, y_1; x_1 - \bar{\lambda} F V_x, y_1 - \bar{\lambda} F V_y)\, dx_1 dy_1$$

FOR ABERRATION FREE SYSTEM

$$\mathcal{D}(V_x, V_y) = \iint_{A(V_x, V_y)} \underline{J}_p(x_1, y_1; x_1 - \bar{\lambda} F V_x, y_1 - \bar{\lambda} F V_y)\, dx_1 dy_1$$

NOTE: THE MAXIMUM SPATIAL FREQUENCY IS

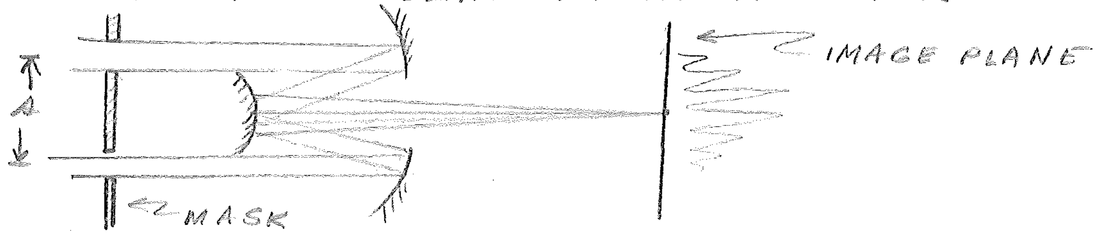
$$\rho_{MAX} = \sqrt{V_x^2 + V_y^2}\,\Big|_{MAX} = 2r_p / \bar{\lambda} F$$

E. GATHERING IMAGE INFORMATION WITH INTERFEROMETERS

FOR INCOHERENT IMAGE, VAN CITTERT-ZERNIKE SAYS WE CAN FIND IMAGE WITH KNOWLEDGE OF $\mu(\Delta x, \Delta y)$.

WILL CONCENTRATE HERE ONLY ON CERTAIN OBJECT INFORMATION

1. THE FIZEAU STELLAR INTERFEROMETER



NEAR THE FRINGES' CENTER, $\mathcal{V}(s) = |\mu_{12}(\Delta)|$

FOR A CIRCULAR SOURCE OF RADIUS a:

$$\mu_{12}(\bar{\Delta}) = 2\, J_1\left(\frac{2\pi a}{\lambda z}\sqrt{\Delta_x^2+\Delta_y^2}\right) \Big/ \left(\frac{2\pi a}{\lambda z}\sqrt{\Delta_x^2+\Delta_y^2}\right)$$

IN TERMS OF <u>ANGULAR DIAMETER</u> $\Theta = 2a/z$

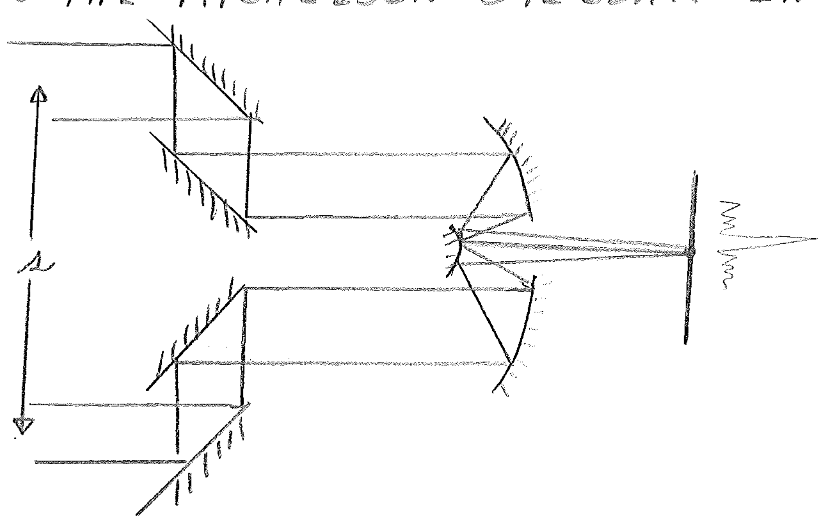
$$\mu_{12}(\bar{\Delta}) = 2\, J_1\left(\frac{\pi\Theta}{\lambda}\sqrt{\Delta_x^2+\Delta_y^2}\right) \Big/ \left(\frac{\pi\Theta}{\lambda}\sqrt{\Delta_x^2+\Delta_y^2}\right)$$

THE FRINGES WILL VANISH WHEN $\Delta_0 = \frac{1.22\,\bar{\lambda}}{\Theta}$

THUS, WE MAY FIND ANGULAR DIAMETER: $\Theta = 1.22\,\bar{\lambda}/\Delta_0$

(NOTE: A PROBLEM IS LIMITATION OF TELESCOPE APERTURE)

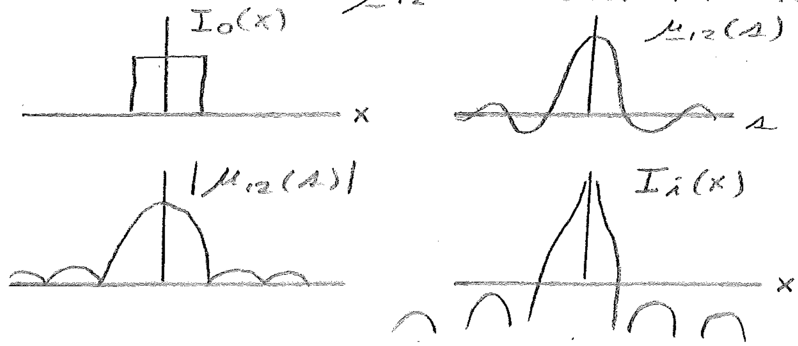
2. THE MICHELSON STELLAR INTERFEROMETER



OVERCOMES PROBLEM'S ASSOCIATED WITH THE FIZEAU INTERFEROMETER

3. INFORMATION OBTAINABLE FROM $|\mu_{12}|$

THE PHASE OF μ_{12} IS VERY IMPORTANT:



IF WE KNOW $\mu_{12}(x)$ IS POS & REAL, WE'RE OKAY

NOTE: $|\mu_{12}(s)|^2$, UPON FOURIER TRANSFORMATION, BECOMES $I \star I$

4. THE PHASE PROBLEM OF COHERENCE THEORY

IF $I_0(x)$ HAS FINITE SUPPORT, & IS BOUNDED, THEN $\mu_{12}(s)$ IS ANALYTIC. SHIFT $I_0(x)$ SO THAT $I_0(x) = 0$ FOR $X < 0$. THEN $\mu_{12}(s)$ BECOMES AN ANALYTIC SIGNAL.

LET $\mu_{12}(s) = e^{\alpha_{12}(s) + j\beta_{12}(s)}$

$\alpha_{12}(s) = \ln|\mu_{12}(s)|$ IS MEASUREABLE

IF $\mu_{12}(s)$ HAS NO ZEROS IN THE LOWER HALF PLANE, THEN $\ln \mu_{12}(s)$ IS ANALYTIC AND

$$\beta_{12}(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln|\mu_{12}(s')| / s' - s \, ds'$$

ie, THEY ARE RELATED BY A <u>HILBERT</u> <u>TRANSFORM</u>

IF ZEROS OF μ_{12} ARE IN THE LOWER HALF PLANE

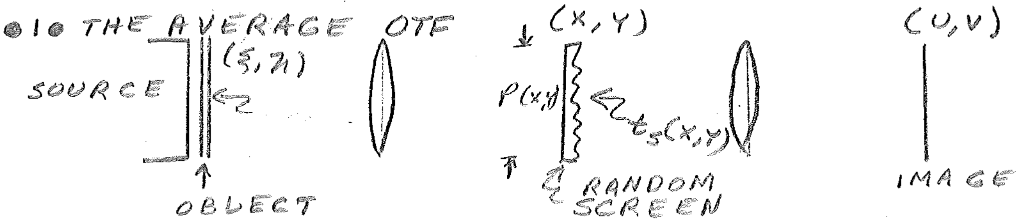
$$\beta_{12}(s) = \frac{1}{\pi} \int \frac{\ln|\mu_{12}(s')|}{s' - s} ds' + \sum_j \arg\left(\frac{s - s_j}{s - s_j^*}\right)$$

IV. IMAGING THROUGH RANDOMLY INHOMOGENEOUS MEDIA

CONSIDER ONLY INCOHERENT SOURCES AND QUASI MONOCHROMATIC LIGHT.

A. EFFECTS OF THIN RANDOM SCREENS

1. THE AVERAGE OTF



WISH TO COMPUTE ENSEMBLE AVERAGE PROPERTIES OF SYSTEM

$\underline{P}(x,y)$ DENOTES (POSSIBLY COMPLEX) PUPIL FUNCTION

THE OTF OR OPTICAL TRANSFER FUNCTION IS

$$\underline{\mathcal{H}}(\nu_x,\nu_y)=\frac{\iint_{-\infty}^{\infty} P(x,y)\,t_A(x,y)\,P^*(x-\bar{\lambda}F\nu_x,\,y-\bar{\lambda}F\nu_y)\,t_A^*(x-\bar{\lambda}F\nu_x\ldots)}{\iint_{-\infty}^{\infty}|\underline{P}(x,y)\,t_A(x,y)|^2\,dx\,dy}$$

$$\bar{\underline{\mathcal{H}}}(\nu_x,\nu_y)=E[\text{NUMERATOR}]/E[\text{DENOMINATOR}]$$

(FOR A PHASE SCREEN, $\bar{\underline{\mathcal{H}}}(\nu_x,\nu_y)=E[\underline{\mathcal{H}}(\nu_x,\nu_y)]$

$$E[\text{NUM}]=\iint_{-\infty}^{\infty}\underline{P}(x,y)\,P^*(x-\bar{\lambda}F\nu_x,\,y-\bar{\lambda}F\nu_y)$$
$$\times\,E\left[t_A(x,y)\,t_A^*(x-\bar{\lambda}F\nu_x,\,y-\bar{\lambda}F\nu_y)\right]dx\,dy$$

ASSUMING SPATIAL STATIONARITY

$$E[t_A(x,y)\,t_A^*(x-\bar{\lambda}F\nu_x,\,y-\bar{\lambda}F\nu_y)]=\underline{J}_T(\bar{\lambda}F\nu_x,\bar{\lambda}F\nu_y)$$

$$\Rightarrow E[\text{NUM}]=\underline{J}_T(\bar{\lambda}F\nu_x,\bar{\lambda}F\nu_y)\iint_{-\infty}^{\infty}P(x,y)\,P^*(x-\bar{\lambda}F\nu_x,\,y-\bar{\lambda}F\nu_y)\,dx\,dy$$

$$E[\text{DEN}]=\underline{J}_T(0,0)\iint_{-\infty}^{\infty}|P(x,y)|^2\,dx\,dy$$

THIS ALL BOILS DOWN TO

$$\bar{\underline{\mathcal{H}}}(\nu_x,\nu_y)=\underline{\mathcal{H}}_o(\nu_x,\nu_y)\,\bar{\underline{\mu}}_t(\bar{\lambda}F\nu_x,\bar{\lambda}F\nu_y)$$

WHERE

$$\underline{\mathcal{H}}_o(\nu_x,\nu_y)=\frac{\iint_{-\infty}^{\infty}P(x,y)\,P^*(x-\bar{\lambda}F\nu_x,\,y-\bar{\lambda}F\nu_y)\,dx\,dy}{\iint_{-\infty}^{\infty}|P(x,y)|^2\,dx\,dy}$$

IS THE OTF OF THE SYSTEM WITHOUT THE SCREEN, AND $\bar{\underline{\mu}}_t(\bar{\lambda}F\nu_x,\bar{\lambda}F\nu_y)=\underline{J}(\bar{\lambda}F\nu_x,\bar{\lambda}F\nu_y)/\underline{J}_T(0,0)$ IS THE OTF OF THE SCREEN ITSELF

THE AVERAGE POINT-SPREAD FUNCTION IS

$$\bar{s}(u,v)=\mathcal{F}^{-1}[\bar{\underline{\mathcal{H}}}(\nu_x,\nu_y)]$$
$$=s_o(u,v)*\bar{s}_t(u,v)$$

• 2 • A RANDOM ABSORBING SCREEN

LET $t_A(x,y) = t_0 + r(x,y)$; $0 \le t_A \le 1$, $E[r]=0$

$$\Rightarrow \underline{J}_T(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = E\left[(t_0 + r(x,y))(t_0 + r(x-\bar\lambda F\nu_x, y-\bar\lambda F\nu_y))\right]$$

$$= t_0^2 + J_r(\lambda F\nu_x, \bar\lambda F\nu_y)$$

$$\Rightarrow \bar\mu_t(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = \left[t_0^2 + J_r(\bar\lambda F\nu_x, \bar\lambda F\nu_y)\right] / (t_0^2 + \sigma_r^2)$$

WHERE $\sigma_r^2 = \overline{r^2}$

EQUIVALENTLY: $\bar\mu_t(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = \frac{t_0^2}{t_0^2+\sigma_r^2} + \frac{\sigma_r^2}{t_0^2+\sigma_r^2}\bar\mu(\bar\lambda F\nu_x, \bar\lambda F\nu_y)$

$$\bar{\mathcal{H}}(\nu_x,\nu_y) = \frac{t_0^2}{t_0^2+\sigma^2}\mathcal{H}_0(\nu_x,\nu_y) + \frac{\sigma_r^2}{t_0^2+\sigma_r^2}\mathcal{H}_0(\nu_x,\nu_y)\,\bar\mu_r(\bar\lambda F\nu_x, \bar\lambda F\nu_y)$$

NO SCREEN   WITH SCREEN

• 3 • A RANDOM PHASE SCREEN

LET $t_S(x,y) = e^{j\phi(x,y)}$, $\phi(x,y)$ IS ZERO MEAN STATIONARY

$$\underline{J}_t(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = E\left[e^{j\phi(x,y) - j\phi(x-\bar\lambda F\nu_x, y-\bar\lambda F\nu_y)}\right]$$

$$= M_\phi(1, -1)$$

$M_\phi(\omega_1,\omega_2)$ IS SECOND ORDER CHARACTERISTIC FUNCTION OF

$$\phi_1 = \phi(x,y), \quad \phi_2 = \phi(x-\bar\lambda F\nu_x, y-\bar\lambda F\nu_y)$$

ASSUME ϕ IS GAUSSIAN

$$\Rightarrow M_\phi(\omega_1,\omega_2) = e^{-\frac{1}{2}[\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + 2\sigma_1\sigma_2\omega_1\omega_2]}$$

$$\sigma_1^2 = \overline{\phi_1^2}, \quad \sigma_2^2 = \overline{\phi_2^2}, \quad \bar\mu_\phi = \overline{\phi_1\phi_2}/\sigma_1\sigma_2$$

THE AVERAGE OTF OF THE SCREEN IS

$$\bar\mu_t(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = e^{-\sigma_\phi^2(1-\bar\mu_\phi(\bar\lambda F\nu_x, \bar\lambda F\nu_y))}$$

WHERE $\sigma_1^2 = \sigma_2^2 = \sigma_\phi^2$ HAS BEIN ASSUMED

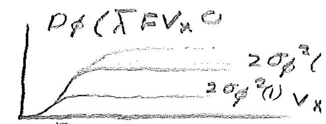
OR, USING THE <u>STRUCTURE</u> <u>FUNCTION</u>

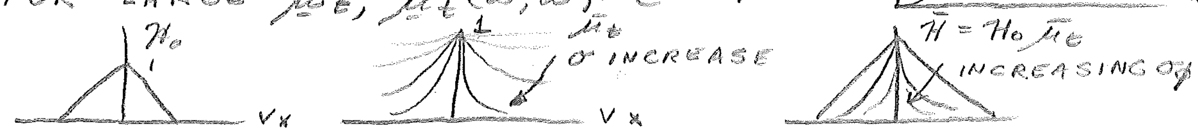
$$D_\phi(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = E\left[\left(\phi(x,y) - \phi(x-\bar\lambda F\nu_x, y-\bar\lambda F\nu_y)\right)^2\right]$$

GIVES $\bar\mu_t(\bar\lambda F\nu_x, \bar\lambda F\nu_y) = e^{-\frac{1}{2}D_\phi(\bar\lambda F\nu_x, \bar\lambda F\nu_y)}$

IT TURNS OUT $D_\phi = 2\sigma_\phi^2[1-\bar\mu_\phi]$

FOR LARGE μ_t, $\bar\mu_t(\infty,\infty) = e^{-\sigma_\phi^2}$





• 4• LIMITING FORMS OF THE OTF AND POINT-SPREAD FUNCTION FOR LARGE PHASE VARIANCE

ASSUME $\sigma_\phi^2 >> 1$ AND $\mu_\phi(\Delta) \simeq 1 - K\Delta^2$

$\Rightarrow \Delta^2 = \Delta_x^2 + \Delta_Y^2$

RECALL $\mu_t(\bar\lambda F v_x, \bar\lambda F v_Y) = e^{-\sigma_\phi^2[1-\bar\mu_\phi(\bar\lambda F v_x, \bar\lambda F v_Y)]}$

IF $\sigma_\phi^2 >> 1$, $\bar\mu_t(\bar\lambda F v) \simeq e^{-K\sigma_\phi^2 \bar\lambda^2 F^2 v^2}$

AND, WE HAVE A GAUSSIAN

$$\Rightarrow \bar{\mathcal{H}}(v_x, v_Y) = \mathcal{H}_0(v_x, v_Y)\,\bar\mu_t(\bar\lambda F v_x, \bar\lambda F v_Y)$$
$$\simeq \bar\mu_t(\bar\lambda F v_x, \bar\lambda F v_Y)$$
$$= e^{-K\sigma_\phi^2 \bar\lambda^2 F^2 v^2}$$

NOTE: POINT SPREAD FUNCTION WILL ALSO BE GAUSSIAN

B. EFFECTS OF AN EXTENDED RANDOMLY INHOMOGENEOUS MEDIA

1. NOTATION AND DEFINITIONS

$n(\bar{r},t) = n_0 + n_1(\bar{r},t) \Leftarrow$ REFRACTIVE INDEX OF ATMOSPHERE

IS MEDIA IS <u>HOMOGENEOUS</u> (SPATIALLY STATIONARY IN 3D):

$$\Gamma_n(\bar{r}) = n_1(\bar{r}_1)\, n(\bar{r}_1 + \bar{r})$$

$$\Phi_n(\bar{\kappa}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \Gamma_n(\bar{r})\, e^{i\bar{\kappa}\cdot\bar{r}}\, d\bar{r} \Leftarrow \text{PWR SPECTRAL DEN}$$

$$\Gamma_n(\bar{r}) = \iiint_{-\infty}^{\infty} \Phi_n(\bar{\kappa})\, e^{-i\bar{\kappa}\cdot\bar{r}}\, d\bar{\kappa}$$

IF MEDIA IS <u>ISOTROPIC</u> (HAS SPHERICAL SYMMETRY)

$$\Phi_n(\kappa) = \frac{1}{2\pi^2\kappa} \int_0^{\infty} \Gamma_n(r)\, r \sin \kappa r \, dr$$

$$\Gamma_n(r) = \frac{4\pi}{r} \int_0^{\infty} \Phi_n(\kappa)\, \kappa \sin \kappa r \, d\kappa$$

WE WILL WISH TO LOOK AT A TWO-DIMENSIONAL SLICE:

$$F_n(\kappa_x, \kappa_y; z) = \int_{-\infty}^{\infty} \Phi_n(\kappa_x, \kappa_y, \kappa_z) \cos \kappa_z z \, d\kappa_z$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} B_n(\bar{\rho}; z)\, e^{i\bar{\kappa}\cdot\rho}\, d\rho$$

$$B_n(\bar{\rho}; z) = \iint_{-\infty}^{\infty} F_n(\bar{\kappa}; z)\, e^{-i\bar{\kappa}\cdot\rho}\, d\kappa$$

WHERE $\bar{\kappa} = (\kappa_x, \kappa_y)$, $\bar{\rho} = (\rho_x, \rho_y)$

$$B_n(\bar{\rho}, z) = E\left[n_1(\bar{\rho}_1, z)\, n_1(\bar{\rho}_1 + \rho; z) \right]$$

IF THE FLUCTUATIONS OF n ARE ISOTROPIC IN Z PLANE:

$$F_n(\kappa; z) = \frac{1}{2\pi} \int_0^{\infty} B_n(\rho; z)\, J_0(\kappa\rho)\, \rho \, d\rho$$

$$B_n(\rho; z) = 2\pi \int_0^{\infty} F_n(\kappa; z)\, J_0(\kappa\rho)\, \kappa \, d\kappa$$

• 2 • ATMOSPHERIC MODEL

<u>TURBULONS</u> = "CLUMPS" OF AIR, EACH WITH REFRACTIVE INDEX

SCALE SIZE OF TURBULONS IS $L = 2\pi/K$

<u>INTERNAL RANGE</u> $L < 10\,mm$ (USE TURBULANCE THEORY)

<u>OUTER SCALE</u> $L \simeq$ HT. ABOVE GROUND

TATARSKI'S REFRACTIVE INDEX SPECTRUM

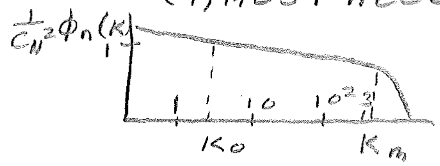
$\Phi_n(K) = 0.033\, C_N^2 K^{-11/3} e^{-K/K_m}$

(1) $K_m \ell_0 = 5.92$

(2) POOR FOR $K \lesssim K_0$

(3) GOOD FOR $K \gtrsim K_m$

(4) MOST ACCURATE FOR $K_0 \leq K \leq K_m$



$C_n^2 \Rightarrow$ <u>STRUCTURE CONSTANT</u> $\rightarrow$ MEASURES TURBULENCE

$10^{-13} m^{-2/3}$ (STRONG) $10^{-17} m^{-2/3}$ (WEAK) $10^{-15} m^{-2/3}$ (TYPICAL)

SINGULARITY DOES EXIST @ $K = 0 \Rightarrow$ NO AUTOCORRELATION

USE, INSTEAD, THE STRUCTURE FUNCTION

$$D_n(r) = 8\pi \int_0^\infty \left(1 - \frac{\sin kr}{kr}\right) \Phi_n(K) K^2 dK$$

$$= C_n^2 r^{2/3} \quad ; \ell_0 < r < L_0$$

• 3 • PLANE WAVE PROPAGATION. THE RYTOV APPROXIMATION

NO SOURCES $\Rightarrow \nabla^2 \bar{E} + k^2 n^2 \bar{E} + 2\nabla(\bar{E} \cdot \bar{\nabla} \ln n) = 0$

IN THE OPTICAL REGION OF SPECTRUM, $\ell_0 \gg \lambda$

$\Rightarrow \nabla^2 E + k^2 n^2 E = 0$

RYTOV METHOD: LET $E = e^{\psi}$. THIS GIVES <u>RICCATI EQ</u>:

$\Rightarrow \nabla^2 \psi(\bar{r}) + \nabla\psi(r) \cdot \nabla\psi(r) + k^2 n^2(r) = 0$

$\psi = \chi + j S$. $\chi = \ln A$ IS THE <u>LOG AMPLITUDE</u>

FOR UNIT PLANE WAVE OF ∞ EXTENT, PERTURBATION THEORY:

$$\psi_1(x,y,z) \simeq \frac{K^2}{2\pi} \int_0^L dz' \iint_{-\infty}^{\infty} dx'dy'\, n(x',y',z') \;;$$
$$\times \exp\left[j k \frac{(x-x')^2 + (y-y')^2}{2(L-z')}\right] \Big/ (L-z')$$

L = PATH LENGTH. FRESNEL DIFFRACTION ASSUMED.

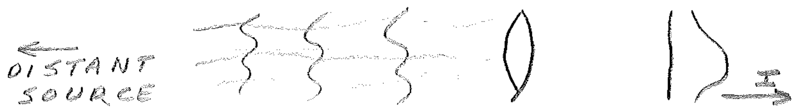
ψ MAY BE ARGUED GAUSSIAN

$\Rightarrow I = A^2 = e^{2\chi}$ IS <u>LOG NORMAL</u>

• 4 • LONG EXPOSURE OTF IN TERMS OF WAVE STRUCTURE FUNCTION

SHORT EXPOSURE: $T \ll \frac{1}{100}$ SEC LONG EXPOSURE: $T \gg \frac{1}{100}$ SEC.

ERGODICITY ASSUMED



INCIDENT ON THE LENS IS $\underline{U}(x,y) = e^{\chi(x,y) + j S(x,y)}$

χ AND S (BY RYTOV APPROXIMATION) ARE GAUSSIAN

OTF IS: $\bar{\mathcal{H}}(\nu_x,\nu_y) = \mathcal{H}_0(\nu_x,\nu_y)\, \underline{J}_A(\bar{\lambda} F \nu_x, \bar{\lambda} F \nu_y) / \underline{J}_A(0,0)$

$$\underline{J}_A(\bar{\lambda} F \nu_x, \bar{\lambda} F \nu_y) = E\left[U(x,y)\, U^*(x - \bar{\lambda} F \nu_x, y - \bar{\lambda} F \nu_y)\right]$$

ASSUME $\underline{J}_A(0,0) = 1$ (i.e., ATMOSPHERE IS LOSSLESS)

ADOPT NOTATION: $\chi_1 = \chi(x,y)$ $\chi_2 = \chi(x - \bar{\lambda} F \nu_x, y - \bar{\lambda} F \nu_y)$

$S_1 = S(x,y)$ $S_2 = S(x - \bar{\lambda} F \nu_x, y - \bar{\lambda} F \nu_y)$

THEN $\underline{J}_A(\bar{\lambda} F \nu_x, \bar{\lambda} F \nu_y) = E\left[e^{\chi_1 + \chi_2 + j S_1 - j S_2}\right]$

SINCE $\overline{\chi_1 S_1} = \overline{\chi_2 S_2}$ AND $\overline{\chi_2 S_1} = \overline{\chi_1 S_2}$, $\overline{(\chi_1 + \chi_2)(S_1 - S_2)} = 0$

THAT IS, χ & S ARE UNCORRELATED, & THUS INDEPENDENT

$$\Rightarrow \underline{J}_A = \overline{e^{\chi_1 + \chi_2}}\ \overline{e^{j(S_1 - S_2)}}$$

BUT $\overline{e^{j(S_1 - S_2)}} = e^{-\frac{1}{2} D_S(\bar{\lambda} F \nu_x, \bar{\lambda} F \nu_y)}$

WHERE $D_S = \overline{(S_1 - S_2)^2}$

$$\overline{e^{\chi_1 + \chi_2}} = e^{\tilde{\Gamma}_\chi(\infty) + \tilde{\Gamma}_\chi(\bar{\lambda} F \nu_x, \bar{\lambda} F \nu_y)}\ e^{2\bar{\chi}}$$

$$\Rightarrow \tilde{\Gamma}_\chi(\Delta_x, \Delta_y) = E\left[\{\chi(x,y) - \bar{\chi}\}(\chi(x - \Delta_x, y - \Delta_y) - \bar{\chi})\right]$$

$= \underline{\text{AUTOCOVARIANCE}}$

ENERGY CONSERVATION SAYS $\bar{I} = 1 = \overline{e^{2\chi}}$

$$\bar{\chi} = -\tilde{\Gamma}_\chi(0) \Rightarrow \overline{e^{\chi_1 + \chi_2}} = e^{\tilde{\Gamma}_\chi(0) + \tilde{\Gamma}_\chi(\bar{\lambda} F \nu) - 2\tilde{\Gamma}_\chi(0)}$$

$$= \exp -\tfrac{1}{2} D_\chi(\bar{\lambda} F \nu)$$

WHERE $D_\chi = \overline{(\chi_1 - \chi_2)^2}$

WE CONCLUDE THAT

$$\underline{J}_A(\bar{\lambda} F \nu) = \underline{\mu}_A(\bar{\lambda} F \nu) = e^{-\frac{1}{2} D(\bar{\lambda} F \nu)}$$

AND

$$\bar{\mathcal{H}}(\nu) = \mathcal{H}_0(\nu)\, e^{-\frac{1}{2} D(\bar{\lambda} F \nu)}$$

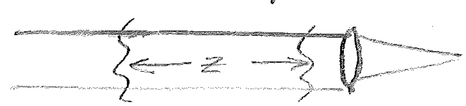
• 5 • NEAR-FIELD CALCULATIONS OF LONG-EXPOSURE OTF

ASSUME (1) OBJECT IS ∞ DISTANCE FROM LENS

(2) TURBULANCE XTENDS DISTANCE z IN FRONT OF LENS & IS HOMO & ISOTROPIC

(3) ISOPLANATIC ASSUMPTION

(4) SYSTEM LIES DEEP IN NEAR FIELD OF TURBULONS (ie, WE JUST HAVE A PHASE DELAY)



$S_1 = \bar{k}\int_0^z n_1(\bar{r}_1)\,dz \qquad S_2 = \bar{k}\int_0^z n_1(\bar{r}_2)\,dz$

$\chi_1 = \chi_2 = \bar{\chi}\ , \ D_\chi(\Delta) = 0$

$n(\vec{r}_1) = n_1(z, s) \qquad n_1(\bar{r}_2) = n_1(z, 0)$

$\bar{\mu}_A(\bar{\lambda} F v) = e^{-\frac{1}{2} D_S(\bar{\lambda} F v)} = e^{-\frac{1}{2}\overline{(S_1 - S_2)^2}}$

$= e^{-\bar{k}^2 C_N^2 \int_0^z (z - \Delta z)\left[(\Delta z^2 + (\bar{\lambda} F v^2))^{\frac{1}{2}} - \Delta z^{2/3}\right] dz}$

WHERE $D_n(r) = C_N^2 r^{2/3} \qquad \ell_0 < r < L_0$

FURTHER SIMPLIFYING GIVES

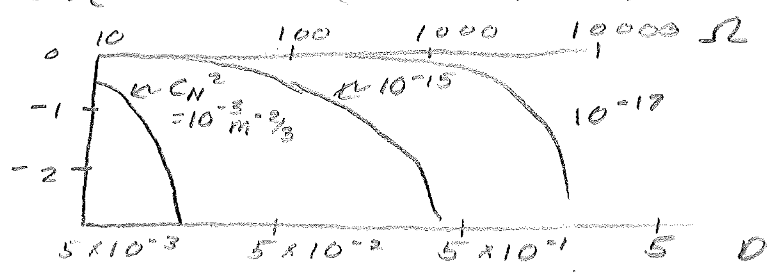
$\bar{\mu}_A(\bar{\lambda} F v) = \exp\left[-\bar{k}^2 C_N^2 z \int_0^\infty \left[(\Delta z^2 + \Delta^2)^{\frac{1}{3}} - \Delta z^{2/3}\right] d\Delta z\right]$

$= \exp\left[-57.44\, C_N^2 \dfrac{z F^{5/3}}{\bar{\lambda}^{1/3}} v^{5/3}\right]$

$= \exp\left[-57.44\, C_N^2\, z/\bar{\lambda}^{1/3}\, \Omega^{5/3}\right]$

WHERE $\Omega = F v$ IS SPATIAL FREQUENCY

$\Omega_{1/e} = \bar{\lambda}^{1/5} / (57.44\, C_N^2 z)^{3/5}$



6. GENERALIZATIONS AND EXTENSIONS (WITHOUT PROOF)

a. WAVE-STRUCTURE FUNCTION BEYOND THE NEAR FIELD

WHEN LENS IS NOT IN THE NEAR DEEP FIELD.

AMPLITUDE FLUCTUATION ARE <u>SCINTILLATIONS</u>

- IT TURNS OUT THAT THE PREVIOUSLY DERIVED TIME-AVERAGE OTF APPLIES OVER MUCH LARGER DISTANCES THAN HAVE BEEN SUPPOSED FROM THE NEAR FIELD CALCULATIONS

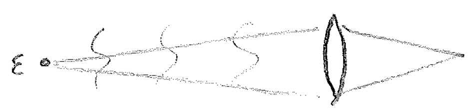
b. EFFECTS OF Z DEPENDENCE OF C_N^2

ASSUME C_N^2 "JUMPS" EVERY ΔZ METERS

TAKING LIMIT, WE GET.

$$\bar{\mu}_A(\bar{\lambda}\Omega) = \exp\left(-\tfrac{1}{2}\, 2.91\, \bar{k}^2 \int_0^z C_N^2(z)\,dz\, (\bar{\lambda}\Omega)^{5/3}\right)$$

c. STRUCTURE FUNCTION FOR A SPHERICAL WAVE



$$D_r(r) = \tfrac{3}{8}\left[2.91\, \bar{k}^2 C_N^2 z\, r^{5/3}\right]$$

SAME AS PLANE WAVE $\times\ 3/8$

d. EFFECTS OF FINITE OUTER SCALE

PREVIOUS $\Phi_n(\kappa) = 0.033\, C_N^2 \kappa^{-11/3} e^{-\kappa^2/\kappa_m^2}$

PREDICTS ∞ RMS & INGNORES OUTER SCALE OF TURBULANCE

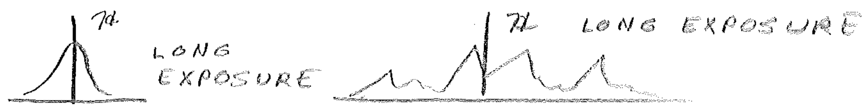
ANOTHER SUGGESTION HAS BEE

$$\Phi_n(\kappa) = 0.063\, \overline{n_1^2}\, L_0^3 \quad \frac{e^{-\kappa^2/\kappa_m^2}}{(1+\kappa^2 L_0^2)^{11/6}}$$

AGREES WELL WITH TATARSKI FOR LARGE WAVE NUMBERS PROVIDED $C_N^2 \doteq 1.9\, \overline{n_1^2}\, L^{-2/3}$

e. SHORT EXPOSURE MTF

MTF = <u>MODULATION TRANSFER FUNCTION</u>



MUST SUBTRACT OUT ATMOSPHERIC TILT.

GIVES "NEAR FIELD" & "FAR FIELD" RESULTS

THERE'S SEVERE PHASE RESPONSE, BUT BETTER HIGH FREQUENCY RESPONSE.

f. LIMITATIONS: RYTOU APPROXIMATION GOOD WHEN

$\overline{\chi^2} \leq 8$ (AMPLITUDE) $\qquad D_S \ll \pi$ (PHASE)

V. STATISTICS OF DETECTION PROCESSES

A. PHOTON COUNTING STATISTICS

1. SEMI-CLASSICAL MODEL FOR PHOTON COUNTING

ASSUMPTIONS (1) FOR SUFFICIENTY SMALL Δt

$$P[1;t;t+\Delta t] = \alpha I(t)\Delta t$$

(2) $P[0;t,t+\Delta t] = 1 - \alpha I(t)\Delta t$

(3) # OF PHOTOEVENTS IN NONOVERLAPPING REGIONS IS INDEPENDENT

(4) $I(t)$ IS DETEMINISTIC

CONSIDER INTERVAL $t+\tau$ TO $t+\tau+\Delta t$

$$P(K;t,t+\tau+\Delta\tau) = P[K;t,t+\tau][1-\alpha I(t+\tau)\Delta\tau] + P[K-1;t,t+\tau]\,\alpha I(t+\tau)\Delta\tau$$

OR $\frac{1}{\Delta\tau}[P(K;t,t+\tau+\Delta\tau) - P(K;t;t+\tau)] = \alpha I(t+\tau) \times [P(K-1;t;t+\tau) - P(K,t;t+\tau)]$

LET $\Delta\tau \to 0 \Rightarrow \frac{dP(K,t,t+\tau)}{d\tau} = \alpha I(t+\tau)[P(K-1;t,t+\tau) - P(K,t,t+\tau)]$

FOR $K=0 \Rightarrow P(0;t;t+\tau) = C_1 \exp[-\alpha\int_t^{t+\tau} I(\xi)d\xi]$

SINCE $P(0,t,t)=1 \Rightarrow$ LET $W_t(\tau) \triangleq \int_t^{t+\tau} I(\xi)d\xi$

AND $P[0,t,t+\tau] = e^{-\alpha W_t(\tau)}$

$K=1$ GIVES $P(1,t;t+\tau) = \alpha W_t(\tau)\, e^{-\alpha W_t(\tau)}$

GOING FURTHER GIVES <u>POISSON</u> <u>DISTRIBUTION</u>

$$P(K,t;t+\tau) = \frac{1}{K!}[\alpha W_t(\tau)]^K e^{-\alpha W_t(\tau)}$$

OR, SINCE WE MUST BE "GIVEN" W:

$$P(K;t;\tau|W) = \frac{[\alpha W_t(\tau)]^K}{K!} e^{-\alpha W_t(\tau)}$$

$$\bar{K} = \alpha W_t(\tau) = \alpha\int_t^{t+\tau} I(\xi)d\xi$$

α COMES FROM ENERGY ON DETECTOR OF AREA A:

$$E = A\int_t^{t+\tau} I(\xi)d\xi$$

WE EXPECT $\bar{K} = \frac{\eta E}{h\bar{\nu}}$ WHERE η IS DETECTOR'S <u>QUANTUM EFFICIENCY</u>

$$\Rightarrow \alpha = \eta A / h\bar{\nu}$$

IN GENERAL $P(K,t;\tau) = \int_0^\infty P(K;t;\tau|W)\, P_W(W)\,dW$

FOR STATIONARY $I(x)$:

$$P(K,\tau) = \int_0^\infty \frac{(\alpha W)^K}{K!} e^{-\alpha W} P_W(W)\,dW \Leftarrow \text{MANDEL'S FORMULA}$$

NOTE: K NEED NOT BE POISSON

• 2 • PHOTOCOUNT STATISTICS FOR WELL-STABALIZED SINGLE-MODE LASER RADIATION

$$W_t(\tau) = \int_t^{t+\tau} I_o \, d\xi = I_o \tau \Rightarrow P_W(W) = \delta(W - I_o \tau)$$

$$P(K;\tau) = \frac{(\bar{K})^K}{K!} e^{-\bar{K}} \quad ; \quad \bar{K} = \alpha I_o \tau$$

THUS, LASER LIGHT IS POISSON

• 3 • PHOTOCOUNT STATISTICS OF POLARIZED THERMAL RADIATION WITH $\tau \ll 1/\Delta\nu$

$$W_t(\tau) = \int_t^{t+\tau} I(\xi)\, d\xi \simeq \tau I(t)$$

RECALL $p_I(I) = \frac{1}{\bar{I}} e^{-I/\bar{I}} \mu(I) \Rightarrow p_W(W) = \frac{1}{\bar{W}} e^{-W/\bar{W}} \mu(W)$

WHERE $\bar{W} = \bar{I}\tau$

GIVES $p(K;\tau) = \frac{1}{1+\bar{K}} \left(\frac{\bar{K}}{1+\bar{K}} \right)^K \quad ; \quad \bar{K} = \alpha \bar{I} \tau$

THIS IS <u>BOSE-EINSTEIN</u> OR <u>GEOMETRIC</u> DISTRIBUTION

• 4 • COMPARISON OF POISSON & BOSE-EINSTEIN DISTRIBUTIONS

	BOSE-EINSTEIN	POISSON	
	MONOTONIC	—	
	$\sigma_K^2 = \bar{K}(1+\bar{K})$	$\sigma_K^2 = \bar{K}$	
	EXCESS FLUCTUATION NOISE		
$\bar{K} \ll 1$	$P(0,\tau) \simeq 1-\bar{K}$	$P(0,\tau) \simeq 1-\bar{K}$	⟸ CLOSE
	$P(1,\tau) \simeq \bar{K}$	$P(1,\tau) = \bar{K}$	⟸ CLOSE

FOR $\bar{K} > 1$, THEY DIFFER A LOT

5. Photocount Statistics for Polarized Thermal Radiation with an Arbitrary Counting Interval

DIVIDE τ INTO m EQUAL SUBDIVISIONS (INDEPENDENT)

CALL SUBDIVISIONS CORRELATION INTERVAL

ASSUME $I(t)$ IS CONSTANT IN EACH OF THESE

WE ARE EQUIVALENTLY DOING A BOXCAR APPROXIMATION:

$$\Rightarrow W = \sum_{i=1}^{m} W_i$$

$$p_{W_i}(w_i) = \frac{1}{\overline{W_i}} e^{-w_i/\overline{W_i}} = \frac{m}{\overline{W}} e^{-m\, w_i/\overline{W}} \quad ; w_i > 0$$

$$\Rightarrow M_{W_i}(\omega) = \frac{1}{1 - j\overline{W_i}\omega} = \frac{1}{1 - j\frac{\overline{W}}{m}\omega}$$

DUE TO STATISTICAL INDEPENDENCE:

$$M_W(\omega) = M_{W_i}^m(\omega) = 1 \Big/ \left[1 - j\frac{\overline{W}}{m}\omega\right]^m$$

INVERSE FOURIER XFORM GIVES GAMMA DISTRIBUTION

$$p_W(w) = \left(\frac{m}{\overline{W}}\right)^m \frac{1}{\Gamma(m)} e^{-m w/\overline{W}}\, w^{m-1} \mu(w)$$

TWO PARAMETERS: m AND $\overline{W} = \overline{I}\tau$. NOTE $\sigma^2 = \frac{\overline{W}^2}{m}$

WE MAY SHOW

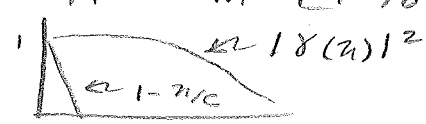
$$\sigma_W^2 = 2\tau \int_0^\tau \left(1 - \frac{\tau}{\tau}\right) \Gamma_I(\tau)\, d\tau - (\overline{W})^2$$

WHERE $\Gamma_I(\xi_1 - \xi_2) = E[I(\xi_1) I(\xi_2)] = (\overline{I})^2 (1 + |\gamma(\tau)|^2)$

$$\sigma_W^2 = \frac{2(\overline{W})^2}{\tau} \int_0^\tau \left(1 - \frac{\tau}{\tau}\right) |\gamma(\tau)|^2 d\tau$$

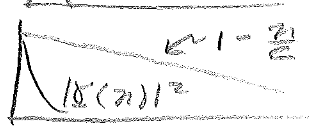
BUT $\sigma_W^2 = \frac{(\overline{W})^2}{m} \Rightarrow m = \left[\frac{2}{\tau}\int_0^\tau \left(1 - \frac{\tau}{\tau}\right)|\gamma(\tau)|^2 d\tau\right]^{-1}$

SHORT τ:



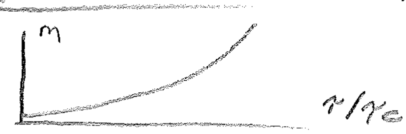
$$m \simeq \left[\frac{2}{\tau}\int_0^\tau \left(1 - \frac{\tau}{\tau}\right) d\tau\right]^{-1} \simeq 1$$

LARGE τ:



$$m \simeq \left[\frac{2}{\tau}\int_0^\infty |\gamma(\tau)|^2 d\tau\right]^{-1} = \frac{\tau}{\tau_c} \simeq \tau/\tau_c$$

IN GENERAL:



AND NOW, THE DISTRIBUTION

$$p(K,\tau) = \int_0^\infty \left[\frac{(\alpha w)^K}{K!} e^{-\alpha w}\right] \frac{(m/\overline{W})^m\, W^{m-1} e^{-m w/\overline{W}}}{\Gamma(m)}\, dw$$

$$= \frac{\Gamma(K+m)}{\Gamma(K+1)\Gamma(m)} \left[1 - \frac{m}{\overline{K}}\right]^K \left[1 + \frac{\overline{K}}{m}\right]^{-m} \Leftarrow \text{NEGATIVE BINOMIAL}$$

$\overline{K} = \alpha\overline{W}$

FOR SMALL τ, THIS BECOMES BOSE-EINSTEIN

(CAUSE $m \simeq 1$)

• 6 • FLUCTUATIONS OF PHOTOCOUNTS - THE DEGENERACY PARAMETER

WE MAY SHOW $\sigma_K^2 = \bar{K} + \alpha^2 \sigma_W^2$

$\bar{K}$ IS PURE SHOT NOISE FROM PHOTOEMMISIVE STATISTICS

RECALL $\sigma_W^2 = \bar{W}^2/m \Rightarrow \sigma_K^2 = \bar{K}(1 + \bar{K}/m)$

$\delta_c = \frac{\bar{K}}{m}$ IS DEGENERACY PARAMETER

(REPRESENT AVE # OF PHOTOEVENTS PER CORREL. INTERVAL)

$\delta_c \ll 1 \Rightarrow$ ALL COUNTS ARE IN SEPARATE CORREL. INTERVLS

$\delta_c \gg \Rightarrow$ "BUNCHED" PHOTOEVENTS. CLASSICALLY INDUCED NOISE PREDOMINATES OVER SHOT NOISE.

$\delta_W = \delta_c/\eta$ = WAVE DEGENERACY PARAMETER

COUNT DISTRIBUTION DETERMINED BY $\bar{K}$ & δ_c:

$$P(K;\tau) = \frac{\Gamma(K + \bar{K}/\delta_c)}{K!\,\Gamma(\bar{K}/\delta_c)} \left[(1+\delta_c)^{\bar{K}/\delta_c} \left(1 + \frac{1}{\delta_c}\right)^{K} \right]^{-1}$$

AS $\delta_c \to 0$, $P(K,\tau)$ GOES TO POISSON

• 7 • SPATIAL COHERENCE EFFECTS

CONSIDER CROSS SPECTRALLY PURE LIGHT. i.e., TEMPORAL AND SPATIAL COHERECE TREATED SEPARATELY

$$m_s \simeq A/A_c \qquad A_c \simeq \bar{\lambda}^2/\Omega \Rightarrow m_s \simeq \frac{A\Omega_0}{\bar{\lambda}^2} \quad (A \gg A_c)$$

$$m_s \simeq 1 \quad (A \ll A_c)$$

THEN

$$\delta_c = \bar{K}/m = \begin{cases} \dfrac{\bar{K}}{\tau\Delta\nu}\,\dfrac{\bar{\lambda}^2}{A\Omega_0} & ;\ \tau \gg \tau_c \quad A \gg A_c \\ \dfrac{\bar{K}}{\tau\Delta\nu} & \tau \gg \tau_c\ ;\ A \ll A_c \end{cases}$$

• 8 • DEGENERACY PARAMETER FOR POLARIZED BLACKBODY RADIATION

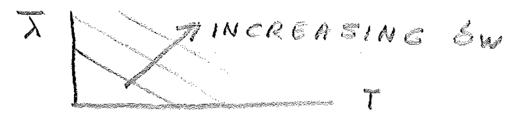
$$E_v = \frac{\Omega_0 A}{\lambda^2} \; h\bar{\nu}\tau \Big/ \left(e^{h\bar{\nu}/kT} - 1\right)$$

Ω_0 = SOLID ANGLE SUBTENDED BY SOURCE

τ = INTEGRATION TIME

GIVES $\delta_c = \dfrac{\bar{K}}{m} = \begin{cases} n \big/ \left(e^{h\bar{\nu}/kT} - 1\right) & ; A >> A_c \\ \dfrac{n \Omega_0 A}{\lambda^2} \big/ \left(e^{h\bar{\nu}/kT} - 1\right) & A << A_c \end{cases}$

CLASSICAL FLUCTUATION MORE IMPORTANT IN MICROWAVES THAN IN OPTICS



• 9 • PHOTOCOUNT STATISTICS FOR PARTIALLY POLARIZED THERMAL LIGHT

$$P_I(I) = \frac{1}{\mathcal{P}\bar{I}} \left[e^{-2I/(1+\mathcal{P})\bar{I}} - e^{-2I/(1-\mathcal{P})\bar{I}} \right] \mu(I)$$

$$\sigma_K^2 = \bar{K}\left[1 + \tfrac{1}{2}(1+\mathcal{P}^2)\bar{K}\right]$$

$$p(K,t) = \frac{1}{\mathcal{P}\bar{K}} \left\{ \left[1 + \frac{2}{(1+\mathcal{P})\bar{K}}\right]^{-K-1} - \left[1 - \frac{2}{(1-\mathcal{P})\bar{K}}\right]^{-K-1} \right\}$$

$\mathcal{P} = 0$ GIVES GAMMA (2M D.OF)

• 10 • THE INVERSION PROBLEM

GIVEN $p(K;\tau) = \displaystyle\int_0^\infty \frac{(\alpha W)^K}{K!} e^{-\alpha W} p_W(W)\, dW$

FIND $p_W(W)$

LET $F(x) \triangleq \displaystyle\int_0^\infty e^{ixW} \left[p_W(W)\, e^{-\alpha W}\right] dW$

SINCE $e^{ixW} = \sum_K \frac{1}{K!} (ixW)^K$

$$F(x) = \sum_{K=0}^{\infty} \left(\frac{ix}{\alpha}\right)^K p(K,\tau)$$

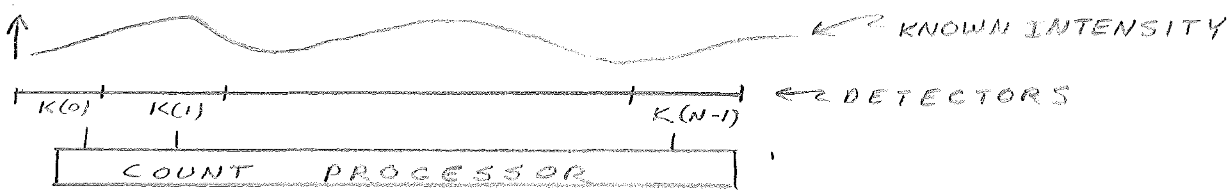
$$\Rightarrow p_W(W) = \frac{1}{2\pi} e^{\alpha W} \int_{-\infty}^{\infty} F(x)\, e^{-ixW}\, dx$$

B. INTERFEROMETRIC MEASUREMENTS AT LOW LIGHT LEVELS

CONSIDER 3 WAYS TO COMPUTE $|\gamma_{12}|$ AND $|\mu_{12}|$

• 1 • PRE-DETECTION OF LIGHT BEAMS

QUASI-MONO LIGHT & ITTY BITTY PINHOLES



INCIDENT ON THE ARRAY IS

$I(x) = I_1 + I_2 + 2\sqrt{I_1 I_2}\, \mu_{12} \cos 2\pi f_0 x + \beta_{12}$

$\mu_{12} = |\mu_{12}|$, $\beta_{12} = \arg \mu_{12}$

ASSUME: 1) FRINGE PERIOD LARGE WRT DETECTOR

2) INTEGRAL # OF FRINGES ACROSS ARRAY

$\bar{K}(\ell) = \alpha\tau \left[I_1 + I_2 + 2\sqrt{I_1 I_2}\, \mu_{12} \cos \frac{2\pi \ell p_0}{N} + \beta_{12} \right.$

$p_0 =$ # PERIODS ON ARRAY , $\alpha = \eta A / h\bar{\nu}$

$\overline{K^2(\ell)} = \bar{K}(\ell) + [\bar{K}(\ell)]^2 (1 + \frac{1}{m})$

$\overline{K(\ell)\, K(q)} = \bar{K}(\ell)\bar{K}(q)\ (1 + \frac{1}{m})$ $\quad \ell \neq q$

PERFORM A <u>DISCRETE</u> <u>FOURIER</u> <u>XFORM</u>:

$X(p_0) = \frac{1}{N} \sum_{\ell=0}^{N-1} K(\ell)\, e^{j 2\pi \ell p_0 / N} = R(p_0) + i\, I(p_0)$

$\bar{R}(p_0) = \alpha\tau \sqrt{I I_2}\, \mu_{12}$ $\quad \bar{I}(p_0) = 0$

$\sigma_R^2 = \frac{2\tau(I_1 + I_2)}{2N} \left[1 + \eta \frac{N}{m} \right]$; $\sigma_I^2 = \frac{\alpha\tau(I_1 + I_2)}{2N}$

$E[(R - \bar{R})(I - \bar{I})] = 0 \Rightarrow R$ & I ARE UNCORRELATED

$\bar{K}_i = \alpha\tau I_i N$

$\left(\frac{S}{N}\right)_{RMS} = \frac{\sqrt{\bar{K}_1 + \bar{K}_2}}{\sqrt{2}}\, \eta = \sqrt{\frac{\bar{K}}{2}}\, \eta$ FOR $\bar{K}_1 = \bar{K}_2 = \bar{K}/2$

TO HOLD FIXED (S/N), $\tau \propto 1/\eta^2$

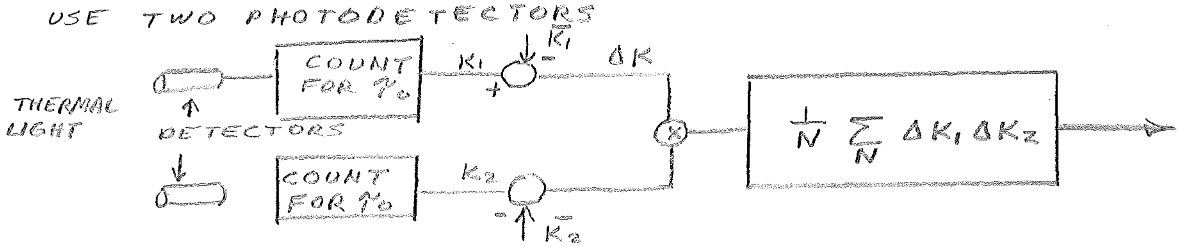
ALSO, FOR $\delta_c = \frac{\bar{K}}{m}$; $m = \frac{\tau}{\tau_c}$, $\left(\frac{S}{N}\right)_{rms} = \sqrt{\frac{\tau}{2\tau_c}} \sqrt{\delta_c}\, \eta$

FOR QUASI THERMAL SOURCES

$(S/N)_{rRM} = \sqrt{\tau / 2\tau_c} \sqrt{\bar{K}/N} \sqrt{\eta}$

2 POST DETECTION CORRELATION OF LIGHT BEAMS
- THE INTENSITY INTERFEROMETER

USE TWO PHOTODETECTORS



$$\overline{\Delta K_1 \Delta K_2} = \frac{\bar{K}_1 \bar{K}_2}{m} \mu_{12}^2$$

PRIMARY NOISE IS SHOT (INDEPENDENT)

TURNS OUT $\left(\frac{S}{N}\right)_{RMS} = \delta_c \gamma^2$ (BAD) $(\delta_c \ll 1,\ \gamma \leq 1)$

INDEPENDENT OF $\tau \Rightarrow$ MAKE IT SHORT

TAKE N COUNT INTERVALS

$$\left(\frac{S}{N}\right)_{RMS} = \sqrt{N}\,\delta_c \gamma_c^2 = \sqrt{\tau/\tau_0}\,\delta_c V^2$$

$\ni$ τ_0 = BASIC COUNT INTERVAL $\tau = N\tau_0$

ADVANTAGES: (1) PATH LENGTH NOT AS IMPORTANT

(2) "ROUGH" MIRRORS CAN BE USED

(3) LITTE EFFECT FROM ATMOSPHERE

DISADVANTAGES: (1) LOW SENSITIVITY

(2) ONLY $|\mu_{12}|$ IS MEASURABLE

Bob Marks

4/26/76

EE 5358 - Oral Talks - Grading Sheet

I. Content (50%) 35

Overall score: 80

II. Presentation (50%) 45

I. **Content**: (Subject; introduction; manner of treating subject; use of math, visual aids in presenting subj; level of presentation, etc.).

Introduction good. might have used a visual aid to introduce & motivate the topic. A little too much detail at front end (we got lost in the detail). You seem to want to give a lecture rather than a 20 minute talk. You did try to be tutorial which was good. I think Di Francia is Italian (he lives in Florence). Summary needs to be stronger.

II. **Presentation**: (Voice modulation, eye contact, organization, visual aids, absence of distracting habits, etc.)

Don't look at your notes so often! eye contact good - visual aids good. Repeated looking at notes shouldn't be necessary; good voice modulation.

ALFRED E. NUEMAN

INTRODUCTION:

- TREAT IMAGING SYSTEM AS
  A COMMUNICATIONS CHANNEL
    OBJECT → TO BE SENT
    IMAGING SYSTEM → CHANNEL (NOISE)
    IMAGE → RECEIVED

DOF A MEASURE OF SYSTEM'S
  INFORMATION CAPACITY

SUMMARY

- CONCEPT OF D.O.F. (DIFRANCIA)
- ATTACK (AND SURVIVAL)
  BY SUPPERRESOLUTION (WOLTER)
- ORIGINAL OBSERVATIONS ON (DIFRANCIA)
  EFFECTS OF COHERENCE
- ATTACK (AND SURVIVAL) (WOLTHER)
- PUPIL OF POINTS APPROACH (GORI)

(COMPUTER)

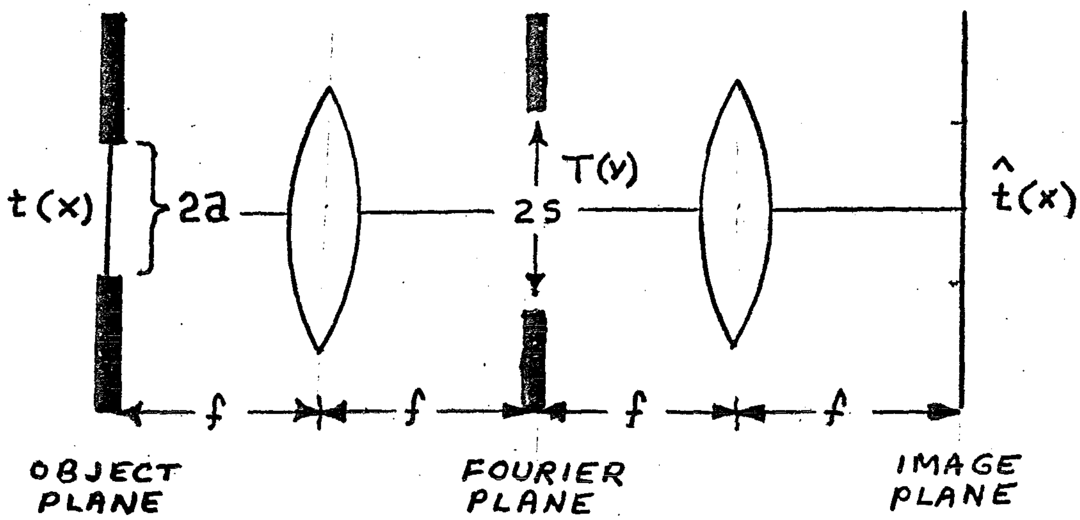
"A LACK OF INFORMATION CANNOT
BE REMEDIED BY ANY MATHEMATICAL
TRICKERY"

C. LANCZOS
<u>LINEAR</u> <u>DIFFERENTIAL</u> <u>OPERATORS</u>
NEW YORK: VAN NOSTRAND, 1961
p. 132

"COPYING THE WORK OF ONE PERSON IS
TERMED 'PLAGIARISM'. COPYING THE
WORK OF MANY IS TERMED 'RESEARCH'"

(PARAPHRASED FROM)
ALFRED. E. NEUMAN
<u>MAD</u> <u>MAGAZINE</u>

DEGREES OF FREEDOM OF AN IMAGE



BANDWIDTH OF IMAGE = $2W = 2s/\lambda f$

$$\hat{t}(x) = \int_{-W}^{W} T(\nu)\, e^{j2\pi\nu x}\, d\nu \qquad ; \quad \nu = \frac{x}{\lambda f}$$

$$= \sum_{n=-\infty}^{\infty} \hat{t}\left(\frac{n}{2W}\right) \mathrm{sinc}(2Wx - n)$$

S = SHANNON NUMBER

= SPACE·BANDWIDTH PRODUCT

= DEGREES OF FREEDOM

$\simeq 4aW = 4as/\lambda f$

$$\hat{t}(x) \simeq \sum_{n=-S/2}^{S/2} \hat{t}\left(\frac{n}{2W}\right) \mathrm{sinc}(2Wx - n)$$

DEGREES OF FREEDOM OF AN IMAGE

SYSTEM / COMMUNICATION ANALOG
COHERENT ILLUMINATION

- OBJECT HAS FINITE SUPPORT SLEPIAN
- OBJECT'S SPECTRUM IS BANDLIMITED
- IMAGE IS BANDLIMITED

EMPLOY SAMPLING THEOREM ON OUTPUT
SAMPLE @ NYQUIST RATE OVER
$-a$ TO $a \Rightarrow S = 4Wa$ # SAMPLES

$\Rightarrow S$ D.O.F.

SUPERRESOLUTION

$\hat{t}(x) =$ IMAGE (OUTPUT)

$t(x) =$ OBJECT (INPUT)

$$\hat{t}(x) = \int_{-a}^{a} t(\xi)\, \mathrm{sinc}\, 2W(x-\xi)\, d\xi$$

- CORRESPONDING INTEGRAL EQ:

$$\lambda_n \psi_n(x) = \int_{-a}^{a} \psi_n(\xi)\, \mathrm{sinc}\, 2W(x-\xi)\, d\xi$$

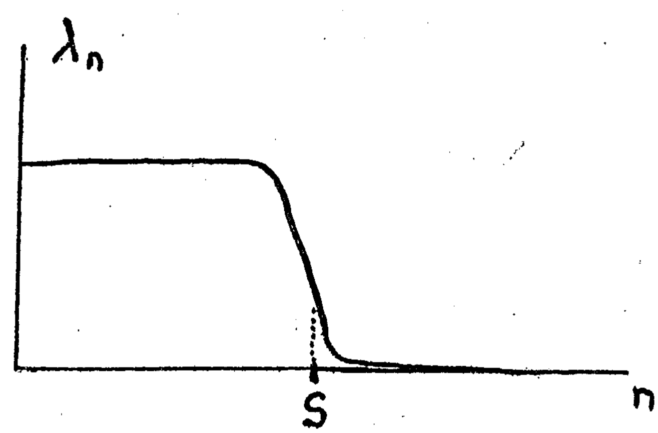
$\psi_n \sim$ PROLATE SPHEROIDAL WAVE FUNCTIONS

- OBJECT: $t(x) = \sum_n c_n \psi_n(x)$

$$c_n = \int_{-\infty}^{\infty} t(x)\, \psi_n^*(x)\, dx$$

- IMAGE: $\hat{t}(x) = \sum_n \lambda_n c_n \psi_n(x)$

$$\lambda_n c_n = \int_{-\infty}^{\infty} \hat{t}(x)\, \psi_n^*(x)\, dx$$



SUPERRESOLUTION

ANALYTICITY OF BANDLIMITED FUNCTION
(TEMES IN FEB. 1973 IEEE PROC)
COMPARE WITH TAYLOR SERIES.

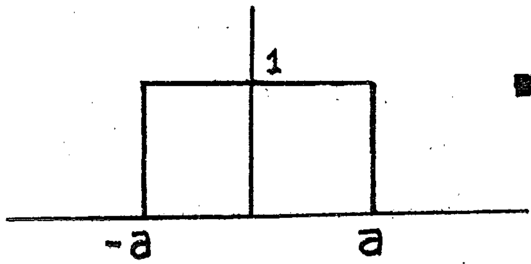
RECTANGULAR PUPIL - SLEPIAN & POLLAK

EIGEN-VALUE'S MESS YOU UP

EFFECTS OF COHERENCE

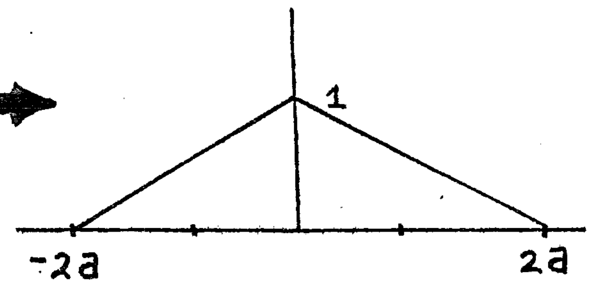
$2\underline{\overline{W}} \rightarrow$ SPATIAL EXTENT OF OBJECT

1. RECTANGULAR PUPIL - ONE DIMENSION



COHERENT

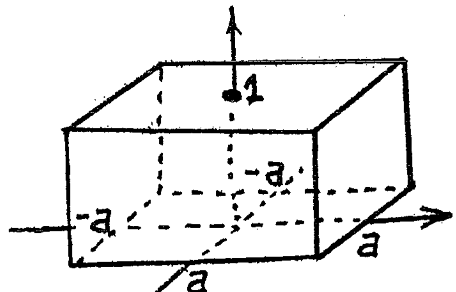
$S_c \cong 8\underline{\overline{W}}a$



INCOHERENT

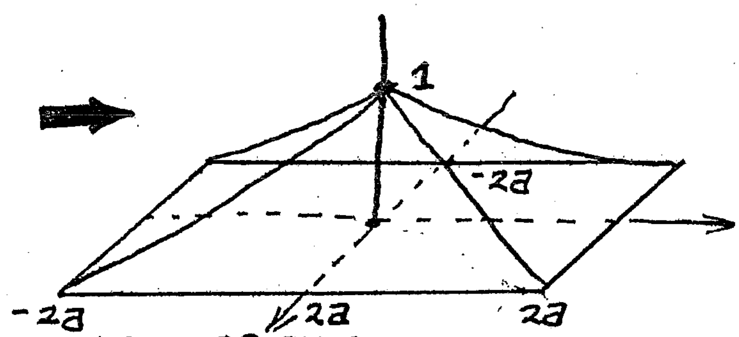
$S_I \cong 8\underline{\overline{W}}a$

2. RECTANGULAR PUPIL - TWO DIMENSIONS



COHERENT

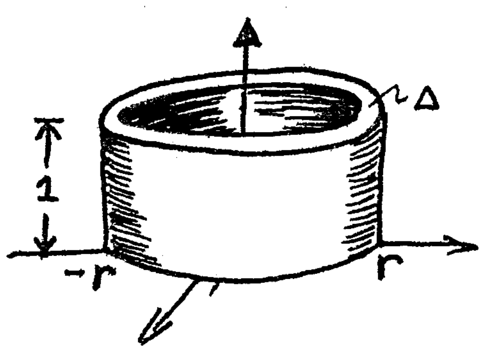
$S_c \cong 32(\underline{\overline{W}}a)^2$



INCOHERENT

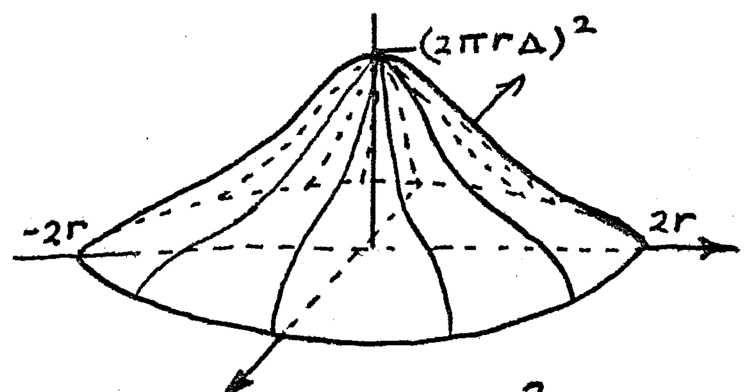
$S_I \cong 64(\underline{\overline{W}}a)^2$

3. THIN RING PUPIL



COHERENT

$S_c \cong 4\pi r\underline{\overline{W}}\Delta$

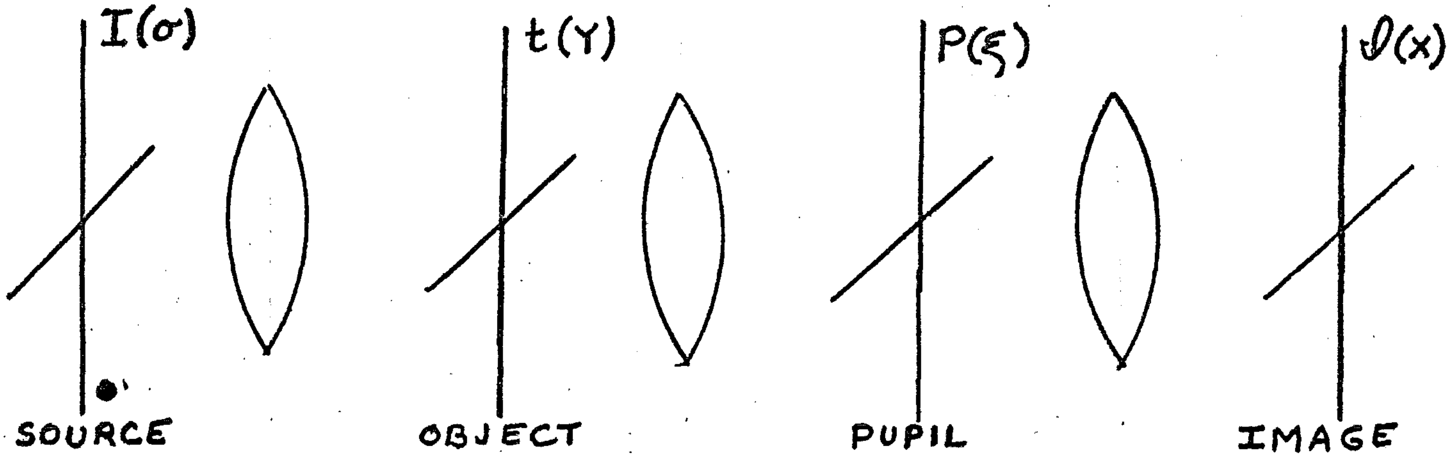


$S_I \cong 16\pi(r\underline{\overline{W}})^2$

EFFECTS OF COHERENCE
DIFRANCIA: DOF & BANDWIDTH

COHERENT → COMPLEX SAMPLES
INCOHERENT → REAL SAMPLES

EFFECTS OF COHERENCE



ASSUME: $P(\xi) = \sum_{\alpha=1}^{n} P_\alpha \, \delta(\xi - \xi_\alpha) \Leftarrow$ n POINTS

$$\mathscr{I}(x) = \mathscr{I}_\alpha + 2\sum_{\alpha>\beta}^{n} \mathscr{I}_{\alpha,\beta} \cos\left[\frac{2\pi}{\lambda f}(\xi_\alpha - \xi_\beta)x + \phi_{\alpha,\beta}\right]$$

$\mathscr{I}_\alpha, \mathscr{I}_{\alpha,\beta}, \phi_{\alpha,\beta}$ FUNCTION OF $I(\sigma), t(Y)$

TOTAL OF $N_L = n(n-1)+1$ D.O.F. (MAX)

- GEOMETRIC EFFICIENCY FACTOR:

  N_{MAX} = # POINTS IN PUPIL'S AUTOCORRELATION

  $\eta = N_{MAX}/N_L$

① COHERENT ILLUMINATION:

$I(\sigma) = \delta(\sigma) \xrightarrow{\text{GIVES}} n(n-1)$ DOF

② INCOHERENT ILLUMINATION

$I(\sigma) = I_0 \xrightarrow{\text{GIVES}} N_{MAX}$ DOF

③ PARTIALLY COHERENT ILLUMINATION

$I(\sigma) = \sum_{\gamma=1}^{m} I_\gamma \, \delta(\sigma - \sigma_\gamma)$

M = # OF POINTS IN CONVOLUTION OF I & P

$\xrightarrow{\text{GIVES}} 2M-1$ DOF

$$\frac{2n^2-2n}{2} \le 2M-1 \le N_{MAX}$$

EFFECTS OF COHERENCE
(GORI & GUITTARI)

TREAT PUPIL AS ARRAY OF POINTS

GEOMETRIC EFFICIENCY FACTOR
N_{MAX} IS MAXIMUM D.O.F. FOR $\eta = 1$.

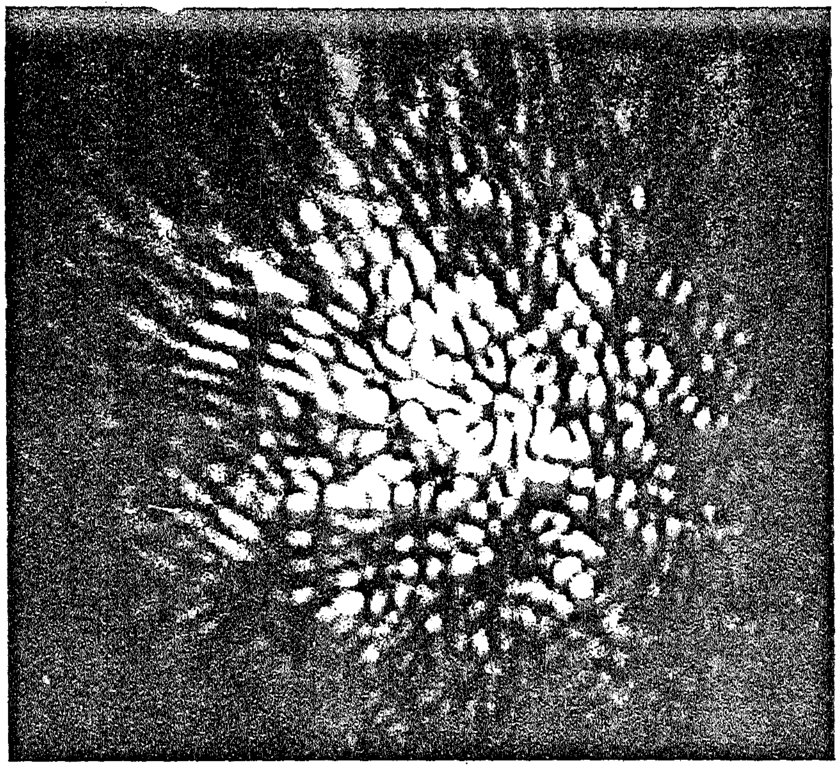


Fig. 7.1. Short exposure narrow band photograph of a magnified image of an unresolved star taken with the 5m Mount Palomar telescope (taken by GEZARI, LABEYRIE and STACHNIK)

the speckle "size" is of the same order of magnitude as the Airy disc of the telescope in the absence of atmospheric turbulence. A long exposure image is simply the sum of many short exposure images, each with a speckle structure that is different in detail, and is therefore a smooth intensity distribution whose diameter is approximately one arc second in good seeing. The minimum speckle size, on the other hand, is approximately 0.02 arc seconds for a 5 m telescope and a mean wavelength of 400nm; by extracting correctly the information in short exposure pictures of objects with more than one resolvable element we can observe detail down to the diffraction limit of the telescope and not be limited to the one arc second of conventional images.

A laboratory simulation illustrating the basic method is shown in Fig. 7.2 for an unresolved star, binary stars of two separations and a resolved star (shown as a uniformly illuminated disc). A large number of short exposure records are taken, each through a different realisation of the

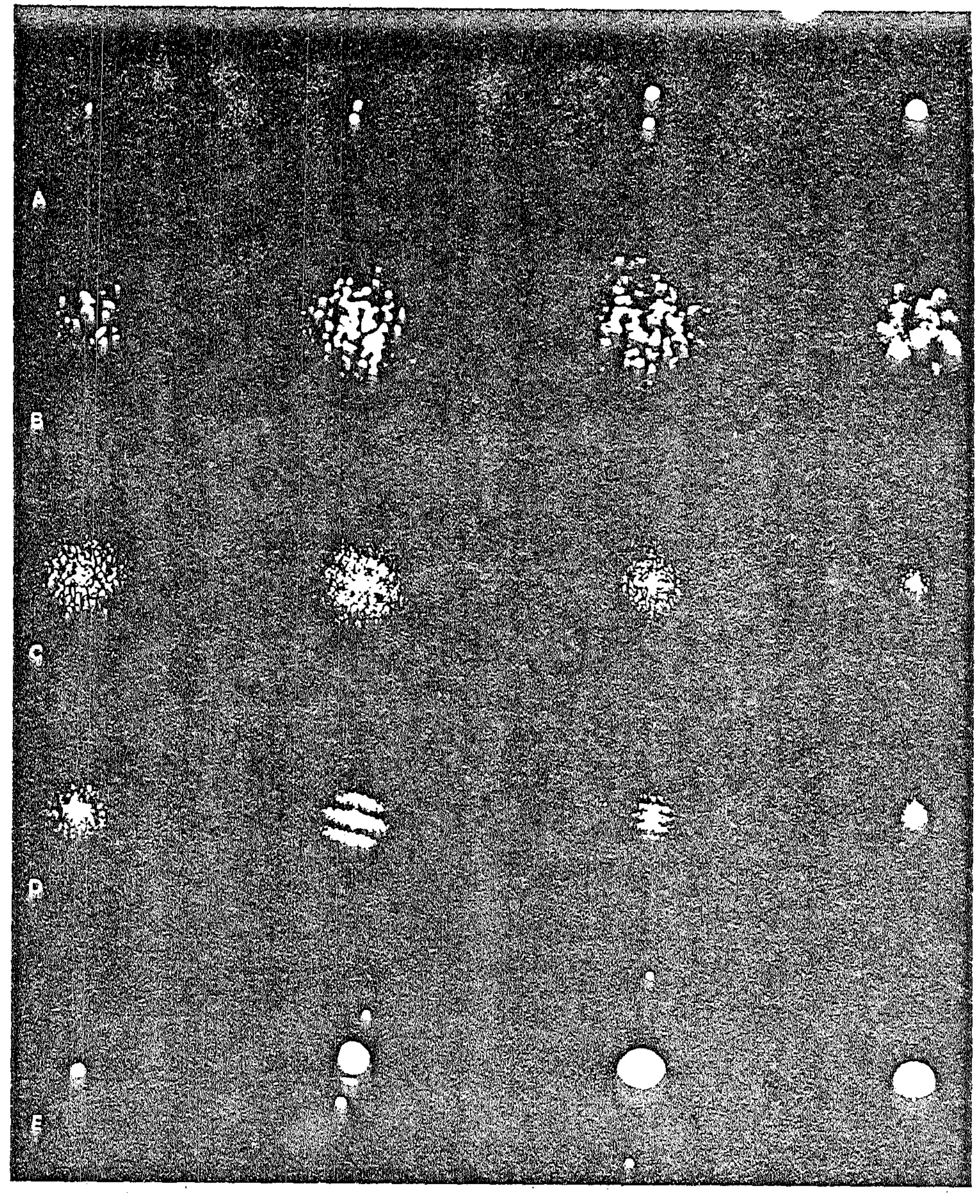


Fig. 7.2. Laboratory simulation showing principles of stellar speckle interferometry. (A, objects; B, typical short exposure photographs; C, diffraction patterns of row B; D, sum of 20 diffraction patterns; E, diffraction pattern of row D) (courtesy of A. LABEYRIE)

Using an analysis similar to that given in Subsection 7.2.1 we can show that if the field entering the telescope from a point source is complex Gaussian,

$$\begin{aligned}\langle T(u',v')\,T^*(u'',v'')\rangle = {} & T_0(u',v')\,T_0^*(u'',v'')\cdot C_A(\xi',\eta')\,C_A^*(\xi'',\eta'') \\ & + \iiiint_{-\infty}^{\infty} C_A(\xi_2-\xi_1,\eta_2-\eta_1) \\ & \cdot C_A^*(\xi_2+\xi''-\xi_1-\xi',\eta_2+\eta''-\eta_1-\eta') \\ & \cdot H_0^*(\xi_1,\eta)\,H_0(\xi_1+\xi',\eta_1+\eta')\,H_0(\xi_2,\eta_2) \\ & \cdot H_0^*(\xi_2+\xi'',\eta_2+\eta'')\,d\xi_1 d\eta_1 d\xi_2 d\eta_2 \end{aligned} \tag{7.22}$$

where $\xi'=\lambda f u'$, etc.

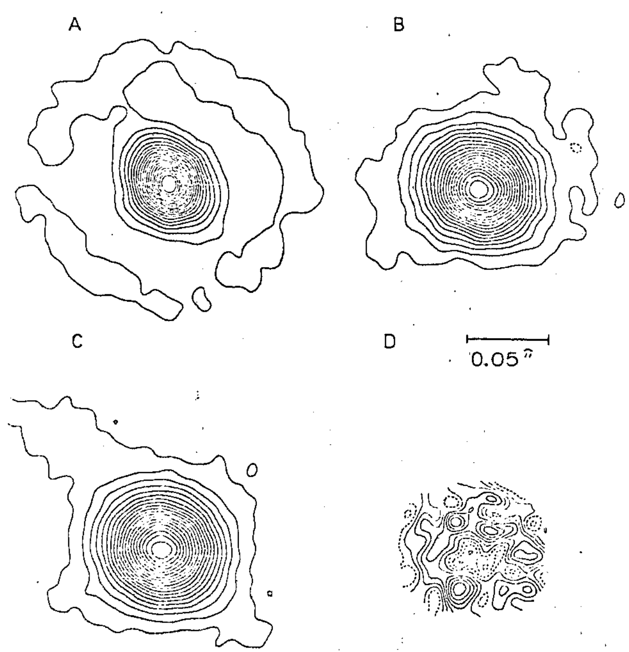


Fig. 7.10A–D. Diffraction-limited images computed from short-exposure photographs by Lynds et. al. [7.29]. A) unresolved star (γ-Ori) apparently showing a diffraction ring, B) Betelgeuse (α-Ori), a red supergiant, in the continuum, C) Betelgeuse in the TiO band, and D) the difference image of B and C. The contour intervals are 5% of the peak intensity in A)–C); in D) the interval is 2% with the dashed curve indicating that the continuum is brighter. The difference picture D) indicates the possible temperature fluctuation over the surface of the star

Examination of (7.22) reveals that the overall transfer function can indeed have non-zero complex values for all (u',v') and (u'',v'') up to the diffraction limit *provided* that the differences $\xi''-\xi'$ and $\eta''-\eta'$ (or $u''-u'$ and $v''-v'$) are small relative to the scale of the seeing correlation function C_A. In practice this means that we must compute phase differences in the spatial frequency domain; the actual phase is then found by summing these differences from the origin to the spatial frequency of interest and clearly such a procedure may lead to the accumulation of errors in the presence of noise. If the seeing is poor (i.e. C_A has non-zero values over relatively small distances) then many phase differences will have to be added to find the phase at a given spatial frequency and consequently any error will be greater than that obtained in good seeing. In the limit of very poor seeing, $C_A \to \delta(\xi)\delta(\eta)$, Eq. (7.22) predicts that diffraction-limited resolution is only obtained if $\xi''-\xi'$ and $\eta''-\eta'$ both equal zero (i.e. no phase information).

Impressive results have been obtained using this technique in a computer simulation with excellent seeing and no measurement noise [7.25]; however, as the authors pointed out, the true test of the technique is on actual short exposure photographs and these results are awaited. It should be noted that the phase information can only be recovered if the telescope guides accurately on the object [7.28].

Another numerical method for finding the original object distribution directly from short exposure photographs has been apparently successfully implemented by Lynds et al. [7.29]. A few bright speckles are selected from short exposure photographs of a resolvable object and are superimposed with the aid of a digital microdensitometer and computer. The resulting picture (see Fig. 7.10) is an image of the object and may contain information to the diffraction limit of the telescope.

To see why this very simple method may provide a picture of the original object, consider first the hypothetical case in which it is supposed that a short exposure photograph of a point source consists of a few widely separated bright speckles. The short exposure image for an extended object is simply the convolution of the object intensity with this speckle pattern (7.1) and provided that the object is not too large this will produce a pattern which is a collection of images of the object, each one centred at an original speckle position. Superposition of these images improves the signal-to-noise ratio. The speckle pattern from a point source is not a collection of widely separated bright speckles; however, because of the approximately negative exponential statistics for the intensity of a speckle pattern from a point source, a few speckles may be significantly brighter than the others and thus give images of a resolved object, albeit on a noisy background. By careful selection of the speckles reliable images may be obtained.

The area that I investigated was the propagation of light in a turbulent atmosphere. The first paper involves the derivation of the mutual coherence and intensity of a gaussian beam on the observation plane after propagating through a turbulent atmosphere. (by T. L. Ho, J. Opt. Soc. Am. 60, 667 (1970). The second paper derives the intensity for a laser beam on the observation plane after propagating through a turbulent atmosphere, and determination of the mean-square value of the phase fluctuations of the turbulence. (by M. Bertolotti, L. Muzii, and D. Sette, J. Opt. Soc. Am. 60, 1603 (1970)).

We consider a random scalar monochromatic wave field each realization of which satisfies the reduced wave equation

$$\nabla V(P) + k^2 n(P) V(P) = 0$$

in a random medium characterized by the random refractive index $n(P)$, where k is the wavenumber. The random medium is assumed to fill the half space $z > 0$ and to have a minimum correlation length l_0 that is much greater than the wavelength λ. The refractive index is assumed to be undergoing small fluctuations.

A gaussian beam is emitted from the plane $z = 0$, propagating toward $z > 0$. The mutual coherence in the observation plane $z = L$ is approximated by

$$\Gamma = \Gamma_0 + \Gamma_{02} + \Gamma_{20} + \Gamma_2 \qquad (1)$$

where

$$\Gamma_0 = E[V_0(P_1) V_0^*(P_2)] \qquad \Gamma_{02} = E[V_0(P_1) V_2^*(P_2)]$$

$$\Gamma_{20} = E\left[\underline{V}_2(P)\,\underline{V}_0^*(P_2)\right] \qquad \Gamma_{11} = E\left[\underline{V}_1(P_1)\,\underline{V}_1^*(P_2)\right]$$

and where $\underline{V}_1$ and $\underline{V}_2$ are first and second order perturbation terms from the solution of the wave equation, and $\underline{V}_0$ is the solution when the refractive index is not fluctuating (1).

$$\underline{V}_1(P) = -\frac{\pi}{\lambda^2}\int_{V'} \frac{\exp[j\frac{2\pi}{\lambda} r(P',P)]}{r(P',P)}\,\epsilon'(P')\,\underline{V}_0(P')\,dP'$$

$$\underline{V}_2(P) = \frac{\pi^2}{\lambda^4}\int_{V'}\int_{V''} \frac{\exp\{j\frac{2\pi}{\lambda}(r(P',P) + r(P'',P'))\}}{r(P',P)\,r(P'',P)}\,\epsilon'(P')$$

$$\cdot\,\underline{V}_0(P'')\,\epsilon'(P'')\,dP'\,dP''$$

where

$\{\}$ represents the ensemble average

$r(P_1,P_2) = \left[(X_1-X_2)^2 + (Y_1-Y_2)^2 + (Z_1-Z_2)^2\right]^{1/2}$

$P_i = (X_i, Y_i, Z_i)$

ϵ' is the small fluctuation in the refractive index

The wave field $\underline{V}_0$ in free space for a guassian beam of effective radius a at $Z = 0$ and radius of wave front curvature r_c is given by (2).

$$\underline{V}_o(P) = \frac{A}{B(z)} \exp\left[-CB^*(z)\frac{X^2+Y^2}{2a^2|B(z)|^2} + j\frac{2\pi}{\lambda}z\right]$$

where

$$B(z) = (1 - z/r_c) + j z\lambda/2\pi a^2$$

$$C = 1 + j2\pi a^2/\lambda r_c$$

and A is the amplitude of the wave at $X=0$

Now Γ_o, Γ_{a}, Γ_{2a}, and Γ_{ii} are calculated individually at the observation plane to obtain Γ. First Γ_o is calculated to be

$$\Gamma_o = \frac{A^2}{|B(L)|^2} \exp\left[-CB^*(L)\frac{X_1^2+Y_1^2}{2a^2|B(L)|^2} - C^*B(L)\frac{X_2^2+Y_2^2}{2a^2|B(L)|^2}\right]$$

The assumptions used to calculate Γ_{a2} are that we have a statistically homogeneous and isotropic medium, the sagittal approximation for an exponential is used (3), the mean-value theorem for the reduced wave equation is used, and the maximum scale ℓ_M of the turbulence that effects the problem is smaller than the propagation length L.

$$\Gamma_{a2} = -\frac{\pi^2}{\lambda^2} L \Gamma_o \int_0^\infty \sigma(s)\, ds$$

Γ_{2a} can be obtained directly from Γ_{a2} because they are complex conjugates. Thus

$$\Gamma_{2a} = -\frac{\pi^2}{\lambda^2} L \Gamma_o \int_0^\infty \sigma(s)\, ds$$

The assumptions to calculate Γ_{ii} are $L/\ell_m \gg 1$, $a/r_c \ll 1$,

$\frac{2\pi a}{\lambda} \gg 1$, $a^2/\lambda L_m \ll 1$ and the sagittal approximation for an exponential is used.

$$\Gamma_{II} = \frac{\pi^2}{\lambda^2} \Gamma_{o} \int_0^L \iint_{-\infty}^{\infty} \tilde{\sigma}(\eta_x, \eta_y, s) \exp\left\{-(\eta_x^2 + \eta_y^2)(L-p)^2 \lambda^2 / 2\pi a^2 |B(L)|^2\right\}$$

$$\cdot \exp\left\{i\left[\beta(p) X_1 - \beta^*(p) X_2\right]\eta_x + i\left[\beta(p) Y_1 - \beta^*(p) Y_2\right]\eta_y\right\}$$

$$\cdot ds\, d\eta_x\, d\eta_y\, dp$$

where

$s = z'' - z'$ $\qquad$ $p = \frac{1}{2}(z'' + z')$

$\beta(p) = B(p)/B(L)$ $\qquad$ $\tilde{\sigma} = \mathcal{F}[\sigma]$

η_x and η_y are the frequency terms of the inverse fourier transform.

Now, by substituting the equations for Γ, Γ_{II}, Γ_{o}, Γ into (1), by restricting our consideration to the special case in which $X_1 = -X_2$ and $Y_1 = -Y_2$, by using the Kolmogorov theory for turbulent atmosphere we have

$$\Gamma \cong \frac{a^2}{|B(L)|^2} \exp\left[-C B^*(L) \frac{X_1^2 + Y_1^2}{2a^2 |B(L)|^2} - C^* B(L) \frac{X_2^2 + Y_2^2}{2a^2 |B(L)|^2}\right]$$

$$\cdot \exp\left[-4.35\, C_n^2 \frac{4\pi^2}{\lambda^2} L \eta_m^{-\frac{5}{3}}\right.$$

$$\left.\cdot \int_0^1 \left\{(1+\theta^2 p^2)^{5/6} {}_1F_1\left[-\frac{5}{6}, 1, -\frac{1}{4} \tilde{\beta}_r^2(p) v_{12}^2 \eta_m^2 / (1+\theta^2 p^2)\right] - 1\right\} dp\right]$$

where

C_n is the structure constant of the refractive index

$$\eta_m = 5.92/\ell_o$$

$$\sigma^2 = L^2 \eta_m^2 \lambda^2 / 4\pi^2 a^2 |B(L)|^2$$

$$\hat{\beta}_m(\rho) = 1 + [\{(1 - L/R)/|B(L)|^2\} - 1]\rho$$

$$|B(L)|^2 = (1 - L/R)^2 + L^2\lambda^2/4\pi^2 a^4$$

$_1F_1$ is the confluent hypergeometric function.

Now, from the mutual coherence we get the average intensity in the observation plane

$$I \cong \frac{A^2}{|B(L)|^2} \exp\left[-\frac{X^2+Y^2}{a^2|B(L)|^2}\right]$$

$$\cdot \exp\left\{4.35\, C_n^2 \frac{4\pi^2}{\lambda^2} L \eta_m^{5/3} \int_0^1 \left[(1+\sigma^2\rho^2)^{5/6} {}_1F_1\left[-\tfrac{5}{6}, 1, \hat{\beta}_m^2(\rho) r^2 \eta_m^2/(1+\sigma^2\rho^2)\right] - 1\right] d\rho\right\}$$

where

$$\hat{\beta}_m(\rho) = -[L\lambda/2\pi a^2 |B(L)|^2]\rho$$

The intensity can be determined by numerical evaluation on a computer.

Now let us consider first a two-beam interferometer, such as the Mach-Zehnder interferometer. When a coherent beam is used, fringes are produced by the interference of rays from conjugate points of the two beams of the interferometer. If one of the two beams travels in a homogeneous medium and the other travels in a medium where the refractive index changes in time and space, the fringe-pattern distortion at each time allows us to study the refractive index in-homogeneities integrated over the path in the arm.

We can write the wave field equations as

$$V_o = A_o \exp[j(\omega t + \phi_o)] \quad \text{for the reference beam}$$

$$V = A(x,y,t) \exp[j(\omega t + \phi(x,y,t))] \quad \text{for the distorted beam}$$

where $A(x,y,t)$ and $\phi(x,y,t)$ are the amplitude and phase of the distorted beam.

Now, if we assume that the two beams incident on the observation plane make a small angle θ between them in the X,Y plane we have

$$I(x,y,t) = |V_o + V|^2 = A_o^2 + A^2(x,y,t) + 2A_o A(x,y,t) \cdot \cos\left[\phi_o - \phi(x,y,t) - \frac{2\pi x \sin\theta}{\lambda}\right]$$

Let us now suppose that $\phi_o - \phi(x,y,t) = \Delta\phi(x,y,t)$, and $A(x,y,t)$ are random variables fluctuating in time, and that we know their joint probability density $p(A, \Delta\phi)$. If a photographic plate is exposed for a length of time to the interference fringes, then we are essentially taking the time average of the instantaneous intensity $I(x,y,t)$. If an ergodic process is assumed we have

$$I(x,y) = A_o^2 + \iint_{-\infty}^{\infty} A^2 + 2A_o A \cos\left[\Delta\phi - \frac{2\pi x \sin\theta}{\lambda}\right] \cdot p(A, \Delta\phi)\, dA\, d\Delta\phi$$

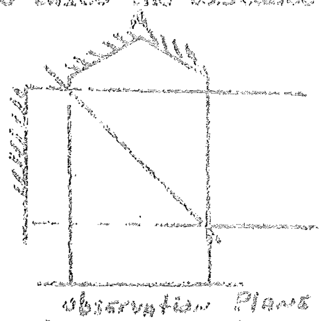
Now assuming A and $\Delta\phi$ are statistically independent and $p(\Delta\phi)$ is gaussian we have

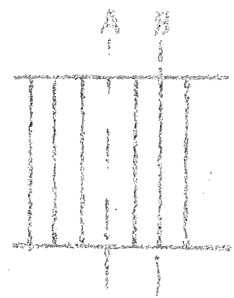
$$I(x,y) = A_o^2 + \langle A^2\rangle + 2A_o\langle A\rangle \cos\left(\frac{2\pi x \sin\theta}{\lambda}\right) \exp\left(-\frac{\sigma^2}{2}\right)$$

and the visibility is

$$V = \frac{2A_o\langle A\rangle e^{-\sigma^2/2}}{A_o^2 + \langle A^2\rangle}$$

Now if we consider the reversing-front interferometer (4) there is no reference beam. With a monochromatic plane wave, fringes are obtained that are parallel to the projection of the roof-prism edge. The fringes are produced by interference of points of the incident beam that are symmetric with respect to the roof-prism edge and that are separated by a distance equal to twice the distance X=AB.





In this case we have for the intensity

$$I(X,t) = |V(X,t) + V(-X,t)|^2$$

$$\simeq A^2(X,t) + A^2(-X,t) + 2A(X,t)A(-X,t)\cos\left[\phi(X,t) - \phi(-X,t) - \frac{2\pi X \sin\theta}{\lambda}\right]$$

and for statistically independent random variables and a gaussian density function for $\Delta\phi$ we have

$$I(X) = 2\langle A^2(X)\rangle + 2\langle A(X)A(-X)\rangle \cos\left(\frac{2\pi X \sin\theta}{\lambda}\right)\exp(-\sigma^2/2) \quad (2)$$

and the visibility is

$$V = \left[\langle A(X)A(-X)\rangle / \langle A^2(X)\rangle\right]\exp(-\sigma^2(X,-X)/2)$$

When amplitude fluctuations are negligible

$$I(x) = 2A^2\left[1 + \cos\left(\frac{2\pi x \sin\theta}{\lambda}\right)\exp\left(-\frac{\sigma^2}{2}\right)\right] \quad (3)$$

and

$$V = \exp\left[-\sigma^2(x_1 - x)/2\right]$$

A measurement of $I(x)$ vs X must therefore give a damped sinusoidal curve whose envelope is the exponential term $\exp(-\sigma^2/2)$. By measuring the visibility we can obtain the variance σ^2 for the gaussian density of $\Delta\phi$.

Two cases of special interest are:

(1) Propagation over short paths. Assume that amplitude fluctuations are negligible. Here we use equation (3).

(2) Propagation over very long paths. It seems reasonable to assume independence between phase and amplitude fluctuations. Here we use equation (2).

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